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Magneto-Fluid Dynamics Division

ANOMALOUS TRANSPORT ARISING FROM NONLINEAR RESISTIVE PRESSURE-DRIVEN MODES IN A PLASMA

Satoshi Hamaguchi

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ABSTRACT

Anomalous transport caused by fluctuations of resistive pressure-driven modes is discussed within the framework of magnetohydrodynamics (MHD). The nonlinear reduced equations describing fluctuations localized near a particular magnetic field line are derived for tokamak and reversed-field-pinch (RFP) plasmas, taking into account nonzero viscosity and heat conductivity. Based on the reduced equations, the relationship between the dynamo effect, or the electric field caused by the fluctuations, and the anomalous heat transport is obtained. For an ideally stable but resistively slightly unstable plasma, the anomalous transport is caused particularly by convective motions. The convection is studied as bifurcation from the linearly unstable equilibrium and the expression of the anomalous transport in a tokamak plasma is obtained as a function of the mean pressure gradient near the critical point. In order to evaluate the effects of the convection on the anomalous transport under various conditions, the reduced equations are also solved numerically. It is found that the Nusselt number, that is, the ratio of the total heat conductivity including the anomalous heat transport to the classical collisional heat conductivity, is significantly large under some conditions. This partially explains the large heat losses in controlled thermonuclear fusion devices.

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I. INTRODUCTION

In a magnetically confined plasma, various types of fluctuations affect the global behavior of the plasma through enhancement of the transport of mass, heat and magnetic fields. The collisions of ions and electrons always generate some diffusion in plasmas, which is called the collisional diffusion. However the transport caused by the fluctuations, which we call the anomalous transport, has in many cases similar or even greater effects on the plasma than the collisional diffusion. The anomalous transport, therefore, has been a subject of much interest in recent years.

In this thesis, we will consider "low-level" or "weak" fluctuations or the fluctuations with small amplitude which vary on faster time scales and smaller spacial scales than the mean fields in a plasma. The fluctuations commonly observed in well confined plasmas are always weak, typically 1% of magnetic field fluctuations in tokamaks and 10% in reversed field pinches (RFP's). The steady convection or the saturated islands on a rational magnetic surface and plasma turbulence are examples of the weak fluctuations that we are particularly interested in.

We will exploit the weakness of the fluctuations in a plasma in order to derive the set of equations of the fluctuations.[1] For general turbulence theories in fluid dynamics, many interesting problems are associated with reasonably strong fluctuations. In such cases, it is known that the system of equations describing fluctuating quantities would not be closed without further approximation (such as the closure method[2]). The model of weak fluctuations to be employed in this thesis, however, enables us to circumvent these difficulties. In this model, introducing a consistent ordering of every physical quantity, we

are able to close the system to the lowest order. For example, the nonlinear term in Faraday's law for the perturbed magnetic field $\underset{\sim}{B}_1$ is $\nabla\times(\underset{\sim}{v}_1\times\underset{\sim}{B}_1 - \langle\underset{\sim}{v}_1\times\underset{\sim}{b}_1\rangle)$ where $\underset{\sim}{v}_1$ is the perturbed velocity field. Since the derivative of term $\underset{\sim}{v}_1\times\underset{\sim}{B}_1$ is much larger than the derivative of its average $\langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle$, where $\langle\ \rangle$ denotes the average of the fluctuations which smoothes away the large derivatives of the fluctuating quantities, the term $\nabla\times\langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle$ does not appear in the equation for $\underset{\sim}{B}_1$ to the lowest order. This procedure significantly simplifies the problem.

One of the most intriguing problems which is believed to be related to the weak fluctuations in a magnetically confined plasma is the spontaneous reversal of the toroidal magnetic field in RFP experiments.[3] Taylor suggested[4] that a slightly resistive plasma minimizes its energy through a magnetic reconnection process, subject to the constraint that the total magnetic helicity of the plasma be conserved. Although he did not specify any particular dynamical process of the plasma relaxation, the predicted minimum energy state agrees well with the experimentally observed equilibrium state of an RFP plasma. Recently it has been demonstrated[1,5,6] that the dynamical description of a plasma based on the nonlinear resistive MHD equations with the presence of weak fluid fluctuations successfully describes the relaxation of a plasma to a state similar to the one predicted by Taylor.

Another interesting phenomenon associated with the weak fluctuations is the anomalous heat transport.[7] For instance, turbulence in an RFP and steady convection on some rational magnetic surfaces in a tokamak enhance the heat conduction across the magnetic flux surfaces

and deteriorate energy confinement of the plasmas. Thus it is important to understand how the anomalous heat transport behaves for a given set of macroscopic profiles of the plasma, such as the pressure profile, in order to determine the energy confinement properties of the plasma. Determination of the dependence of the anomalous heat transport on such macroscopic conditions is one of the main goals of this thesis.

The class of the weak fluctuations with which we are particularly concerned here is the one generated by the resistive g-mode or, more precisely, the resistive fast interchange mode,[8] which is the instability caused by pressure or temperature gradients acting against the curvature of the magnetic field lines in a plasma with finite resistivity, analogous to the thermal instability causing the Benard convection in a fluid heated from below in a gravity field. These resistive pressure driven modes are expected to be present in tokamaks and RFPs. Recently, Hameiri[7] discussed the turbulent heat conduction without specifying any particular modes and derived some of its general properties. In this thesis we will consider the anomalous heat transport specifically caused by the resistive g-mode fluctuations and we will determine a more precise characterization of the transport.

For the resistive g-mode, the free energy source is the mean pressure (or temperature) gradient, and the energy sink is the collisional diffusion. Therefore when the mean pressure gradient exceeds a critical value, modes localized on a rational surface begin to grow. These modes, however, eventually saturate with finite amplitude if the mean pressure gradient is not too large compared to the critical value, and we have steady convection within a boundary

layer which is analogous to the Benard convection in fluid dynamics. This resemblance of the resistive g-mode to the Benard convection was first pointed out and studied by Dagazian[9] based on a quasi-linear theory[10] with the shape assumption or the assumption that in a slightly nonlinear regime the mode retains its linear shape with a finite amplitude. We will deal here with the similar problem of the convection arising from the resistive g-mode using, however, a different method. Our approach is to use nonlinear bifurcation analysis[11] to derive the dependence of the anomalous heat transport on the mean pressure gradient analytically. In this method, the set of the nonlinear equations of the fluctuations is expanded in terms of the small amplitude, and the complete algorithm to determine all the higher order terms is obtained. Therefore we do not need any further assumptions such as the shape assumption. We will also show the results of numerical computations of the nonlinear equations of the fluctuations, which determine the dependence of the anomalous heat transport on the mean quantities under various conditions.

This thesis consists of nine sections. Section II reviews and extends the set of equations governing the mean quantities obtained by Hameiri.[7] In Section III, we derive the nonlinear reduced equations describing the fluctuations due to the resistive g-modes with finite viscosity and heat conductivity, for both an RFP and a tokamak. Sections IV and V present general discussions on the anomalous transport in an RFP and a tokamak, respectively. Since the set of reduced equations for a tokamak is simpler and easier to treat, Section VI discusses its nonlinear stability, from which we see the similarity of the resistive g-mode and the Benard convection. Section VII

contains the bifurcation analysis of steady convection of a tokamak plasma. Here the dependence of the anomalous transport due to the convection on the mean pressure gradient is derived near the critical pressure gradient. Section VIII summarizes the aforementioned properties of the anomalous transport with some quantitative results obtained from the numerical simulations. Finally, Section IX contains discussions and conclusions.

II. EQUATIONS OF THE MEAN MOTION

We begin by reviewing the derivation of the set of equations describing the evolution of the mean quantities. Here we will also generalize the earlier work[7] by taking into account the parallel viscosity[12] in addition to the other diffusion coefficients.

We start from the following resistive MHD equations:

$$\rho\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla\right)\underset{\sim}{v} = -\nabla p + \underset{\sim}{J}\times\underset{\sim}{B} - \nabla\cdot\underset{\approx}{\Pi} \qquad (2\text{-}1a)$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla\times(\eta\underset{\sim}{J} - \underset{\sim}{v}\times\underset{\sim}{B}) = 0 \qquad (2\text{-}1b)$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underset{\sim}{v}) = 0 \qquad (2\text{-}1c)$$

$$\frac{\partial p}{\partial t} + \underset{\sim}{v}\cdot\nabla p + \gamma p\nabla\cdot\underset{\sim}{v} = (\gamma-1)(\eta\underset{\sim}{J}\cdot\underset{\sim}{J} + \nabla\cdot(\kappa\nabla T) - \underset{\approx}{\Pi}:\nabla\underset{\sim}{v}) \;, \qquad (2\text{-}1d)$$

where

$$p = \rho T \qquad (2\text{-}1e)$$

$$\Pi = -3\mu_{||}(\hat{\underset{\sim}{b}}\hat{\underset{\sim}{b}} - \frac{1}{3}\underset{\approx}{I})\lambda - \mu_{\perp}\underset{\approx}{\sigma} \qquad (2\text{-}1f)$$

$$\lambda = \hat{\underset{\sim}{b}}\cdot[(\hat{\underset{\sim}{b}}\cdot\nabla)\underset{\sim}{v}] - \frac{1}{3}\nabla\cdot\underset{\sim}{v} \qquad (2\text{-}1g)$$

$$\hat{\underset{\sim}{b}} = \underset{\sim}{B}/|\underset{\sim}{B}| \qquad (2\text{-}1h)$$

In Eqs. (2-1), symbols have their usual meanings: ρ, p, T and $\underset{\sim}{v}$ are the mass density, pressure, temperature and velocity, respectively. $\underset{\sim}{B}$ is the magnetic field, satisfying $\nabla\cdot\underset{\sim}{B} = 0$, $\underset{\sim}{J} = \nabla\times\underset{\sim}{B}$ is the current density, γ is the ratio of the specific heats, η is the resistivity tensor, and κ is the heat conductivity tensor. Both η and κ are

assumed to be positive definite. Equation (2-1f) is the definition of the stress tensor $\underline{\underline{\Pi}}$, where $\underline{\underline{I}}$ denotes the unit tensor and $\underline{\underline{\sigma}}$ is the rate-of-strain tensor defined by

$$\sigma_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\nabla\cdot\underline{v}$$

for the (i,j) component in Cartesian coordinates. Here δ_{ij} indicates Kronecker's delta, that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. $\nabla\cdot\underline{\underline{\Pi}} = \sum_i \partial\Pi_{ij}/\partial x_i$ is the divergence of the tensor Π and $\Pi:\nabla\underline{v}$ is the contraction defined as $\Pi:\nabla\underline{v} = \sum_i \sum_j \Pi_{ij}\partial v_i/\partial x_j$. It follows that

$$\underline{\underline{\Pi}}:\nabla\underline{v} = -3\mu_{\parallel}\lambda^2 - \frac{\mu_{\perp}}{2}\,\mathrm{tr}\,\underline{\underline{\sigma}}^2 ,$$

where tr denotes the trace of the tensor.

We assume that the plasma is confined in either a toroidal or a cylindrical vessel with a perfectly conducting wall. Therefore we impose the boundary conditions

$$\hat{\underline{n}}\cdot\underline{B} = 0$$

$$\underline{v} = 0$$

and

$$\underline{E}\times\hat{\underline{n}} = 0 ,$$

where $\hat{\underline{n}}$ denotes the unit vector normal to the boundary, and $\underline{E}$ denotes the electric field, defined by

$$\underset{\sim}{E} = \eta \underset{\sim}{J} - \underset{\sim}{v} \times \underset{\sim}{B} .$$

In case of a cylindrical vessel, we identify the two ends of the cylinder and require periodicity of all physical variables.

Equations (2-1a) and (2-1d) may be written in the equivalent form:

$$\frac{\partial}{\partial t} (\rho \underset{\sim}{v}) + \nabla \cdot (\rho \underset{\sim}{v} \underset{\sim}{v}) = -\nabla p + \underset{\sim}{\nabla} \times \underset{\sim}{B} - \nabla \cdot \underset{\approx}{\Pi} \tag{2-2}$$

$$\frac{\partial}{\partial t} (\rho s) + \nabla \cdot (\rho s \underset{\sim}{v}) = \frac{1}{T} \eta \underset{\sim}{J} \cdot \underset{\sim}{J} + \nabla \cdot \left(\frac{1}{T} \kappa \nabla T\right) + \frac{1}{T^2} \kappa \nabla T \cdot \nabla T$$

$$+ \frac{3}{T} \mu_{||} \lambda^2 + \frac{1}{2T} \mu_{\perp} \operatorname{tr} \underset{\approx}{\sigma}^2 . \tag{2-3}$$

Here s denotes the specific entropy defined as

$$s = \frac{1}{\gamma - 1} \log (p/\rho^{\gamma}) . \tag{2-4}$$

For convenience, we work with variables measured in two natural units, in which distances are measured in units of the plasma minor radius a, times are measured in units of the Alfven transit time $a\sqrt{\bar{\rho}/\bar{B}}$, where $\bar{B}$ and $\bar{\rho}$ are typical values of the magnitude of the magnetic field and the mass density of the plasma, and the magnetic field and mass densities are measured relative to $\bar{B}$ and $\bar{\rho}$, respectively. Other variables are then measured accordingly.

We now consider the fluctuations in the plasma described by Eqs. (2-1). Since we are interested in evolution of "mean" physical quantities under some influence of turbulence, we have to clarify what

we mean by a "mean" or "averaged" value of each physical quantity. Let us denote $\langle f\rangle$ the averaged value of a physical quantity f. Then we require that the averaging operator $\langle\ \rangle$ satisfy the so-called Reynolds conditions[2]:

$$\langle f+g\rangle = \langle f\rangle + \langle g\rangle \ , \tag{2-5a}$$

$$\langle af\rangle = a\langle f\rangle \ , \tag{2-5b}$$

$$\langle a\rangle = a \ , \tag{2-5c}$$

$$\langle \frac{\partial f}{\partial x_i}\rangle = \frac{\partial\langle f\rangle}{\partial x_i} \ , \quad (1 \le i \le 3) \ , \quad \langle \frac{\partial f}{\partial t}\rangle = \frac{\partial\langle f\rangle}{\partial t} \ , \tag{2-5d}$$

$$\langle\langle f\rangle g\rangle = \langle f\rangle\langle g\rangle \ , \tag{2-5e}$$

where f and g are some physical quantities and a is a constant. Since we consider low-level fluctuations varying rapidly in space and time superimposed on a mean motion, which varies slowly in space and time, we may think of $\langle\ \rangle$ as either an ensemble average or an average over the small space- and fast time-scales of the fluctuations. However, the following discussion in this section does not depend on the particular choice of the averaging operation $\langle\ \rangle$ so long as it satisfies conditions (2-5). Therefore we do not specify the actual form of the averaging here and will wait until further assumption is made on the fluctuations so that we can easily specify the most appropriate form of averaging.

Once an averaging operator $\langle\ \rangle$ satisfying conditions (2-5) is defined, we can write every physical quantity as the sum of the mean part, denoted by subscript o and the fluctuating part, denoted by subscript 1. This decomposition is unique. For example, the magnetic

field $\underset{\sim}{B} = \underset{\sim}{B}_o + \underset{\sim}{B}_1$, where $\underset{\sim}{B}_o = \langle \underset{\sim}{B} \rangle$ and $\underset{\sim}{B}_1 = \underset{\sim}{B} - \langle \underset{\sim}{B} \rangle$. It is easy to show from properties (2-5) that $\langle B_1 \rangle = 0$.

For simplicity, we take η to be a small scalar quantity. We now derive the dynamical equations of the mean quantities. Since we are dealing with low-level fluid fluctuations, we use the following assumptions: (1) mean quantities, taken to be O(1) except for the mean flow $\underset{\sim}{v}_o$, evolve on a resistive time scale and vary on the minor radius length scale, i.e., $a|\underset{\sim}{v}_o| = O(\eta)$, $a^2 \frac{\partial}{\partial t} = O(\eta)$ and $a|\nabla| = O(1)$; (2) the fluctuations may vary more rapidly in space and time than the mean quantities, i.e., $a^2 \frac{\partial}{\partial t} >> O(\eta)$, $a|\nabla| >> O(1)$; (3) the energy of the fluctuations is considerably smaller than the mean magnetic energy, i.e., $|\underset{\sim}{B}_1|^2$, $\rho_o|\underset{\sim}{v}_1|^2$, p_1, $\rho_o T_1$, $T_o \rho_1$, $<< |\underset{\sim}{B}_o|^2$. We note, therefore, that the fluctuating current $\underset{\sim}{J}_1 = \nabla \times \underset{\sim}{B}_1$ is of the order of $\underset{\sim}{J}_o$, or $a|\underset{\sim}{J}_1|/|\underset{\sim}{B}_o| = O(1)$; (4) the perpendicular viscosity $\mu_\perp$ is taken to be $O(\eta)$ while the parallel viscosity $\mu_{||}$ is taken to be O(1). Here we also only consider the perpendicular component $\kappa_\perp$ and the parallel component $\kappa_{||}$ of the conductivity tensor κ: it is assumed that $\kappa_\perp$ and $\kappa_{||}$ are $O(\eta)$ and O(1), respectively. In general, these diffusion coefficients η, μ and κ depend on such physical variables as ρ, T and $\underset{\sim}{B}$ and, therefore, they also fluctuate. However, from these ordering assumptions, it is easy to check that the fluctuating parts of the diffusion coefficients are so small that they enter neither the equation of the mean motion nor the equation of the fluctuations.

Under these ordering assumptions, we obtain the set of equations for the mean quantities by taking the average of the MHD equations. Averaging (2-2), (2-1b), (2-1c), (2-3), (2-1e) and (2-4) yields, respectively,

$$\nabla p_o = \underset{\sim}{J}_o \times \underset{\sim}{B}_o \tag{2-6a}$$

$$\frac{\partial \underset{\sim}{B}_o}{\partial t} + \nabla\times(\eta\underset{\sim}{J}_o - \underset{\sim}{v}_o\times\underset{\sim}{B}_o - \langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle) = 0 \tag{2-6b}$$

$$\frac{\partial \rho_o}{\partial t} + \nabla\cdot(\rho\underset{\sim}{v}_o + \langle\rho_1\underset{\sim}{v}_1\rangle) = 0 \tag{2-6c}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_o s_o) + \nabla\cdot(\rho_o s_o\underset{\sim}{v}_o + \rho_o\langle s_1\underset{\sim}{v}_1\rangle + s_o\langle\rho_1\underset{\sim}{v}_1\rangle)\\ = \frac{1}{T_o}(\eta\underset{\sim}{J}_o\cdot\underset{\sim}{J}_o + \langle\eta\underset{\sim}{J}_1\cdot\underset{\sim}{J}_1\rangle) + \nabla\cdot\left(\frac{1}{T_o}\kappa\nabla T_o\right) + \frac{1}{T_o}(\kappa\nabla T_o\cdot\nabla T_o + \langle\kappa\nabla T_1\cdot\nabla T_1\rangle)\\ + \frac{3}{T_o}\mu_{||}\langle\lambda_1^2\rangle + \frac{1}{2T_o}\mu_{\perp}\,\mathrm{tr}\,\langle\sigma_1^2\rangle\end{aligned} \tag{2-6d}$$

$$p_o = \rho_o T_o \tag{2-6e}$$

$$s_o = \log\frac{1}{\gamma-1}(p_o/\rho_o^{\gamma}) \; . \tag{2-6f}$$

Here Eqs. (2-6b) and (2-6c) are exact while Eqs. (2-6a), (2-6e) and (2-6f) holds to O(1), Eq. (2-6d) to O(η).

There is an important relationship among the averaged quantities:

$$\begin{aligned}\langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle\cdot\underset{\sim}{J}_o + \frac{1}{(\gamma-1)T_o}\langle p_1\underset{\sim}{v}_1\rangle\cdot\nabla T_o - T_o\langle\rho_1\underset{\sim}{v}_1\rangle\cdot\nabla s_o\\ = -\left(\langle\eta\underset{\sim}{J}_1\cdot\underset{\sim}{J}_1\rangle + \frac{1}{T_o}\langle\kappa\nabla T_1\cdot\nabla T_1\rangle + 3\mu_{||}\langle\lambda_1^2\rangle + \frac{\mu_{\perp}}{2}\,\mathrm{tr}\,\langle\sigma_1^2\rangle\right) + f\end{aligned} \tag{2-7}$$

where

$$f = -\nabla\cdot\langle(p_1 + \underset{\sim}{B}_o\underset{\sim}{B}_1)\underset{\sim}{v}_1 - (\underset{\sim}{v}_1\cdot\underset{\sim}{B}_1)\underset{\sim}{B}_o + \frac{1}{2}\rho_o\underset{\sim}{v}^2\cdot\underset{\sim}{v}\rangle\ . \tag{2-8}$$

This may be derived as follows: the energy equation of the system is found by combining Eqs. (2-1)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\underset{\sim}{B}^2 + \frac{1}{2}\rho\underset{\sim}{v}^2 + \frac{p}{\gamma-1}\right)$$

$$+ \nabla\cdot\left[(\eta\underset{\sim}{J} - \underset{\sim}{v}\times\underset{\sim}{B})\times\underset{\sim}{B} + \frac{1}{2}\rho\underset{\sim}{v}^2\underset{\sim}{v} + \frac{\gamma}{\gamma-1}p\underset{\sim}{v} - \kappa\nabla T - \Pi\underset{\sim}{v}\right] = 0\ .$$

Averaging this equation, we obtain the expression for $\partial\left(B_o^2/2 + p_o/(\gamma-1)\right)/\partial t$ to order of η. We also form another expression for $\partial\left(B_o^2/2 + p_o/(\gamma-1)\right)/\partial t$ from the averaged equations (2-6). Subtracting these two expressions yields Eq. (2-7).

By using the following definitions

$$\underset{\sim}{\varepsilon} = \langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle - \frac{1}{\rho_o}\langle\rho_1\underset{\sim}{v}_1\rangle\times\underset{\sim}{B}_o \tag{2-9}$$

$$\underset{\sim}{u}_o = \underset{\sim}{v}_o + \frac{1}{\rho_o}\langle\rho_1\underset{\sim}{v}_1\rangle\ , \tag{2-10}$$

Equation (2-7) may be written as

$$\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o + \rho_o\langle s_1\underset{\sim}{v}_1\rangle\cdot\nabla T_o$$

$$= -\left\{\langle\eta\underset{\sim}{J}_1\cdot\underset{\sim}{J}_1\rangle + \frac{1}{T_o}\langle\kappa\nabla T_1\cdot\nabla T_1\rangle + 3\mu_{||}\langle\lambda_1^2\rangle + \frac{\mu_\perp}{2}\,\mathrm{tr}\,\langle\sigma_1^2\rangle\right\} + f\ , \tag{2-11}$$

where the relation $s_1 = p_1/p_o - \gamma\rho_1/\rho_o$ was used. Writing Eq. (2-6d) in terms of p_o and using Eq. (2-11), we rewrite Eqs. (2-6):

$$\frac{\partial \underset{\sim}{B}_o}{\partial t} + \nabla\times(\eta \underset{\sim}{J}_o - \underset{\sim}{\varepsilon} - \underset{\sim}{u}_o\times\underset{\sim}{B}_o) = 0 \tag{2-12a}$$

$$\frac{\partial \rho_o}{\partial t} + \nabla\cdot(\rho_o \underset{\sim}{u}_o) = 0 \tag{2-12b}$$

$$\frac{\partial p_o}{\partial t} + \underset{\sim}{u}_o\cdot\nabla p_o + \gamma p_o(\nabla\cdot\underset{\sim}{u}_o)$$

$$= (\gamma-1)\left[\eta \underset{\sim}{J}_o\cdot\underset{\sim}{J}_o - \underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o + \nabla\cdot(\kappa\nabla T_o) - \nabla\cdot(p_o \langle s_1\underset{\sim}{v}_1\rangle) + f\right] \tag{2-12c}$$

From these equations, it is clear that $\underset{\sim}{\varepsilon}$ and $-p_o\langle s_1\underset{\sim}{v}_1\rangle$ are the extra electric field and the extra heat flux caused by the fluctuations, respectively. We therefore call $\underset{\sim}{\varepsilon}$ the anomalous electric field and $-p_o\langle s_1\underset{\sim}{v}_1\rangle$ the anomalous heat flux. We also call the effect of $\underset{\sim}{\varepsilon}$ the dynamo effect since it is related to the generation of the magnetic field.[17]

Together with Eq. (2-6a), Eqs. (2-12) are the closed set of equations to evolve the mean quantities if the anomalous electric field $\underset{\sim}{\varepsilon}$, the anomalous heat flux $-p_o\langle s_1\underset{\sim}{v}_1\rangle$ and the function f are given. We note that f vanishes for localized fluctuations as will be discussed. We also note that the quantity $\underset{\sim}{u}_o$ is determined, as in the Grad-Hogan model[18] for diffusion, by the requirement that all mean quantities evolve through a sequence of magnetostatic equilibrium states satisfying Eq. (2-6a). The dependence of the anomalous electric field $\underset{\sim}{\varepsilon}$ on the mean profiles was determined by Bhattacharjee and Hameiri,[1,6] for the case of tearing-mode-induced turbulence. One of the goals of

this thesis is to determine the anomalous heat flux from the set of equations of the resistive g-mode fluctuations, which will be derived in the next section.

We now briefly comment on some properties of the anomalous electric field $\underset{\sim}{\varepsilon}$ and the anomalous heat flux $-p_o\langle s_1 \underset{\sim}{v}_1\rangle$. By integrating Eq. (2-11) over the total volume of the plasma, we obtain

$$\int (\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o + \rho_o\langle s_1\underset{\sim}{v}_1\rangle\cdot\nabla T_o)d^3x$$

$$= -\int \left(\langle\eta \underset{\sim}{J}_1\cdot\underset{\sim}{J}_1\rangle + \frac{1}{T_o}\langle\kappa\nabla T_1\cdot\nabla T_1\rangle + 3\mu_{||}\langle\lambda_1^2\rangle + \frac{\mu_\perp}{2}\,\mathrm{tr}\,\langle\underset{\approx}{\sigma}_1^2\rangle\right)d^3x\ , \quad (2\text{-}13)$$

since the volume integral of f vanishes. Equation (2-13) gives the relationship of the integrals of $\underset{\sim}{\varepsilon}$ and $\langle s_1\underset{\sim}{v}_1\rangle$ over the total plasma volume. Moreover, in certain limiting cases, the term $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o$ appearing in Eqs. (2-11) and (2-12e), is shown to vanish. In fact, Hameiri proved[7] that $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o = 0$ if the resistivity $\eta = 0$. We will also show in Section V that the term $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o$ vanishes if the pressure gradient ∇p_o is taken to be vanishingly small. In either of these cases, we have, from Eq. (2-13),

$$\int \rho_o\langle s_1\underset{\sim}{v}_1\rangle\cdot\nabla T_o d^3x < 0 \qquad (2\text{-}14)$$

This inequality implies that the anomalous heat flux tends to transport heat in the direction opposite to the mean temperature gradient ∇T_o. This is similar to the collisional heat conductivity. As a special case, if the fluctuations are highly localized and if we may take the integral in a small volume, e.g., in the vicinity of a rational

surface, we expect that $\rho_0\langle s_1 \underset{\sim}{v}_1\rangle\cdot\nabla T_0 < 0$ holds locally. This consideration suggests that, in this case, the anomalous heat flux $-\rho_0\langle s_1 \underset{\sim}{v}_1\rangle$ has a form of $K^2\nabla T_0$, where K^2 is a non-negative function which may depend on mean quantities, including ∇T_0.

In the following sections, we will discuss the fluctuations due to the resistive g-mode and characterize the anomalous heat flux in more detail.

III. REDUCED EQUATIONS OF THE FLUCTUATIONS

We will now derive the equations describing the dynamics of the fluctuating quantities arising from the resistive g-mode. This is an extension of the reduced equations obtained in Ref. 1.

We now assume that the mode is localized along a mean magnetic field line so that the perpendicular derivative of a fluctuating quantity, i.e., the derivative across the field line is of order $1/\delta$, where δ is a small parameter measuring the localization of the mode, while the parallel derivative of a fluctuating quantity, i.e., the derivative along the field line, is of order 1. Assuming that the resistivity η is a scalar constant for simplicity, we take δ to be $O(\sqrt{\eta})$. The time derivative of a fluctuating quantity is assumed to be $O(1)$, while the time derivative of a mean quantity is $O(\eta)$ as mentioned before. All the mean quantities are at most of order 1 except for $\underset{\sim}{v}_o$ which is $O(\delta^2)$, and the fluctuating quantities $\underset{\sim}{v}_1$, $\underset{\sim}{B}_1$, p_1 and T_1 are assumed to be $O(\delta)$. These orderings are consistent with the assumption we made in Section II and agree with the scaling of the resistive fast interchange mode in the linear stability theory.[8]

We now describe the nonlinear mode equations according to these orderings. For simplicity we take the parallel viscosity coefficient $\mu_{||}$ to be smaller that $O(1)$ so that it does not eneter the equation of the fluctuations. The more complete version of the mode equations including the parallel viscosity coefficient $\mu_{||}$ is derived in Appendix A. The perpendicular viscosity is taken to be $O(\delta^2)$, as before. Subtracting Eq. (2-6a) from Eq. (2-1a), we obtain to $O(\delta)$

$$\rho_o\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_1\cdot\nabla\right)\underset{\sim}{v}_1 = -\nabla p_1 + \underset{\sim}{J}_o\times\underset{\sim}{B}_1 + \underset{\sim}{J}_1\times\underset{\sim}{B}_o + \underset{\sim}{J}_1\times\underset{\sim}{B}_1 + \mu_\perp\Delta_\perp\underset{\sim}{v}_1 \; . \qquad (3\text{-}1)$$

Similarly, subtracting Eq. (2-6b) from Eq. (2-1e), we have, again to $O(\delta)$,

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} + \nabla\times(\eta \underset{\sim}{J}_1 - \underset{\sim}{v}_o\times\underset{\sim}{B}_1 - \underset{\sim}{v}_1\times\underset{\sim}{B}_o - \underset{\sim}{v}_1\times\underset{\sim}{B}_1) = 0 . \tag{3-2}$$

We note that, since $\langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle$ is a mean quantity, the term $\nabla\times\langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle$ is smaller than the term $\nabla\times(\underset{\sim}{v}_1\times\underset{\sim}{B}_1)$ and was dropped in Eq. (3-2). Subtracting Eq. (2-6c) from Eq. (2-1c), and Eq. (2-12c) from Eq. (2-1d), yields

$$\frac{\partial \rho_1}{\partial t} + \nabla\cdot(\rho_o\underset{\sim}{v}_1 + \rho_1\underset{\sim}{v}_1) = 0 \tag{3-3}$$

$$\frac{dp_1}{dt} + \underset{\sim}{v}_1\cdot\nabla p_o + \gamma p_o\nabla_\perp\cdot\underset{\sim}{v}_1 = (\gamma-1)\nabla\cdot(\kappa\nabla T_1) \tag{3-4}$$

to the lowest order. In Eq. (3-4), we may write $\kappa\nabla = \kappa_{||}\nabla_{||} + \kappa_\perp\nabla_\perp$, assuming $\kappa_{||} = O(1)$ and $\kappa_\perp = O(\delta^2)$ as before. Here $\nabla_{||}$ and $\nabla_\perp$ denote derivatives parallel and perpendicular to the total magnetic field line $\underset{\sim}{B}_o + \underset{\sim}{B}_1$, respectively. Since $\nabla_\perp$ is the same as the derivative perpendicular to the mean magnetic field $\underset{\sim}{B}_o$ to the lowest order, we actually define $\nabla_\perp$ as $\nabla_\perp = \nabla - \underset{\sim}{b}(\underset{\sim}{B}_o\cdot\nabla)$, where $\underset{\sim}{b} = \underset{\sim}{B}_o/|\underset{\sim}{B}_o|^2$.

By rewriting the right-hand side of Eq. (3-1) using the relations $\underset{\sim}{J}_o = \nabla\times\underset{\sim}{B}_o$ and $\underset{\sim}{J}_1 = \nabla\times\underset{\sim}{B}_1$, we have

$$\rho_o(\frac{\partial}{\partial t} + \underset{\sim}{v}_{1\perp}\cdot\nabla)\underset{\sim}{v}_1 = -\nabla(p_1+\underset{\sim}{B}_o\cdot\underset{\sim}{B}_1) + (\underset{\sim}{B}_o\cdot\nabla)\underset{\sim}{B}_1 + (\underset{\sim}{B}_1\cdot\nabla)\underset{\sim}{B}_o + (\underset{\sim}{B}_{1\perp}\cdot\nabla)\underset{\sim}{B}_1 .$$

The perpendicular component of this equation is

$$\nabla_{\perp}(p_1 + \underset{\sim}{B}_o \cdot \underset{\sim}{B}_1) = 0$$

to the leading order. By assuming the mode to vanish away from the rational surface as it does in the linear theory, we conclude to $O(\delta)$

$$p_1 + \underset{\sim}{B}_o \cdot \underset{\sim}{B}_1 = 0 \ .$$

Therefore, the parallel component of $\underset{\sim}{B}_1$ is proportional to $-p_1$. From the equation $\nabla \cdot \underset{\sim}{B} = 0$ together with the ordering assumptions and the relation $\nabla \cdot \underset{\sim}{B}_o = 0$, we have to $O(1)$

$$\nabla_{\perp} \cdot \underset{\sim}{B}_1 = 0 \ .$$

It follows that there exists a scalar function A of order δ^2 such that

$$\underset{\sim}{B}_1 = \nabla_{\perp} A \times \underset{\sim}{b} - p_1 \underset{\sim}{b} \tag{3-5}$$

Similarly, from Eq. (3-4), we conclude $\nabla_{\perp} \cdot \underset{\sim}{v}_1 = 0$ to $O(1)$. Therefore there exists a scalar function ϕ of order δ^2 such that

$$\underset{\sim}{v}_1 = \nabla_{\perp} \phi \times \underset{\sim}{b} - \nu \underset{\sim}{b} \ , \tag{3-6}$$

where we write $-\nu \underset{\sim}{b}$ as the parallel component of the velocity $\underset{\sim}{v}_1$.

With these new functions, the perpendicular component of Eq. (3-2) becomes

$$\frac{\partial A}{\partial t} + (\nabla_{\perp} A \times \nabla_{\perp} \phi) \underset{\sim}{b} = \eta \Delta_{\perp} A + \underset{\sim}{B}_o \cdot \nabla \phi \ , \tag{3-7}$$

and the parallel component of Eq. (3-1), along with the relation $\underset{\sim}{B}_o \cdot \underset{\sim}{B}_1 = -p_1$, is given as

$$\frac{d}{dt} p_1 - (\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp}) \cdot \nabla \nu - \underset{\sim}{v}_1 \cdot \nabla (p_o + \underset{\sim}{B}_o^2) - \underset{\sim}{B}_o^2 (\nabla_\perp \cdot \underset{\sim}{v}_1) - \eta \Delta_\perp p_1 = 0 , \quad (3\text{-}8)$$

where $d/dt = \partial/\partial t + \underset{\sim}{v}_{1\perp} \cdot \nabla$, and $\Delta_\perp = \nabla_\perp^2$. Operation $\underset{\sim}{B}_o \cdot \nabla \times$ (Eq. (3-1)) yields

$$\rho_o \frac{d}{dt} \Delta_\perp \phi = (\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp}) \cdot \nabla (\Delta_\perp A) - 2\underset{\sim}{b} \times \nabla (p_o + \frac{1}{2} \underset{\sim}{B}_o^2) \cdot \nabla p_1$$

$$- 2\underset{\sim}{b} \cdot \nabla (\frac{1}{2} \underset{\sim}{B}_o^2)(\Delta_\perp A) + \mu_\perp \Delta_\perp^2 \phi \qquad (3\text{-}9)$$

and the parallel component of Eq. (3-1) becomes

$$\rho_o \frac{d}{dt} \nu = (\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp}) \cdot \nabla p_1 + \underset{\sim}{B}_{1\perp} \cdot \nabla p_o + \mu_\perp \Delta_\perp \nu . \qquad (3\text{-}10)$$

By using the relation

$$\frac{\rho_1}{\rho_o} = \frac{p_1}{p_o} - \frac{T_1}{T_o} , \qquad (3\text{-}11)$$

which holds to the lowest order, we eliminate ρ_1 from the system and derive the equations for p_1 and T_1. We eliminate the term $\nabla_\perp \cdot \underset{\sim}{v}_1$ from Eqs. (3-4) and (3-8) and obtain an equation for p_1:

$$\frac{dp_1}{dt} + \underset{\sim}{v}_{1\perp}\cdot\nabla p_o = \frac{\gamma p_o}{\gamma p_o + \underset{\sim}{B}_o^2}\left[2\underset{\sim}{v}_{1\perp}\cdot\nabla\left(p_o + \frac{1}{2}\underset{\sim}{B}_o^2\right) + (\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp})\cdot\nabla\nu_1 + \eta\Delta p_1\right]$$

$$+ \frac{(\gamma-1)\underset{\sim}{B}_o^2}{\gamma p_o + \underset{\sim}{B}_o^2}\,\nabla\cdot\left[(\kappa_{||}\nabla_{||} + \kappa_{\perp}\nabla_{\perp})T_1\right] . \qquad (3\text{-}12)$$

Likewise, from Eqs. (3-3), (3-4) and (3-11), we get an equation for T_1:

$$\frac{d}{dt}T_1 + \underset{\sim}{v}_{1\perp}\cdot\nabla T_o = \frac{(\gamma-1)T_o}{\underset{\sim}{B}_o^2 + \gamma p_o}\left[2\underset{\sim}{v}_{1\perp}\cdot\nabla\left(p_o + \frac{1}{2}\underset{\sim}{B}_o^2\right) + (\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp})\cdot\nabla\nu_1 + \eta\Delta p_1\right]$$

$$+ \frac{(\gamma-1)}{\rho_o}\,\frac{\underset{\sim}{B}_o^2 + p_o}{\gamma p_o + \underset{\sim}{B}_o^2}\,\nabla\cdot\left[(\kappa_{||}\nabla_{||} + \kappa_{\perp}\nabla_{\perp})T_1\right] . \qquad (3\text{-}13)$$

Either Eq. (3-12) or Eq. (3-13) may be replaced by the following entropy equation:

$$\frac{ds_1}{dt} + (\underset{\sim}{v}_{1\perp}\cdot\nabla)s_o = \frac{1}{p_o}\,\nabla\cdot\left[(\kappa_{||}\nabla_{||} + \kappa_{\perp}\nabla_{\perp})T_1\right] , \qquad (3\text{-}14)$$

where we note that, to the lowest order,

$$s_1 = \frac{\gamma}{\gamma-1}\frac{T_1}{T_o} - \frac{p_1}{p_o} . \qquad (3\text{-}15)$$

We now further simplify these equations assuming that the plasma

is confined in a cylinder and all the mean quantities depend only on its radius r. It is convenient to introduce the following independent variables,

$$x = |\sigma|^{1/2}\,\frac{r-r_o}{r_o}\,, \qquad y = \frac{B_\theta}{r_o B}\left(\frac{\sigma}{|\sigma|}\right)|\sigma|^{1/2}(z - \mu(r)\theta)\,,$$

$$\tilde{\theta} = |\sigma|\theta\,, \qquad \tau = \frac{B_\theta|\sigma|}{r_o\sqrt{\rho_o}}\,t\,, \qquad (3\text{-}16)$$

Here (r,θ,z) denote the usual polar coordinates and r_o denotes a particular radius in the vicinity of which we consider the motion of the modes. B_θ and B_z denote the azimuthal and longitudinal components of the mean magnetic field $\underset{\sim}{B}_o$, respectively. We note that the $\tilde{\theta}$-direction (with fixed x and y) is the direction along the equilibrium magnetic field line $\underset{\sim}{B}_o(r)$ and not the azimuthal direction anymore. Furthermore, we define

$$B = |\underset{\sim}{B}_o|\,,$$

$$\mu = \frac{rB_z}{B_\theta}\,, \qquad (3\text{-}17)$$

$$\sigma = \frac{B_\theta}{B}\,\mu'\,,$$

where prime denotes d/dr and every quantity here is evaluated at $r = r_o$.

The multiscale assumption allows us to take every fluctuating quantity to be a function of x, y, $\tilde{\theta}$ and τ as well as r_o,t, while every

mean quantity depends only on r_o and t. From the assumption of the ordering we made, we have

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} = O(\frac{1}{\delta}) ,$$

$$\frac{\partial}{\partial \tau} = O(1) , \qquad \frac{\partial}{\partial \tilde{\theta}} = O(1) , \tag{3-18}$$

$$\frac{\partial}{\partial t} = O(\delta^2) \text{ and } \frac{\partial}{\partial r_o} = O(1) .$$

We thus find that Eqs. (3-7)-(3-10), (3-12) and (3-13) are the nonlinear partial differential equations with independent variables x, y, $\tilde{\theta}$ and τ, and with constant coefficients depending only on parameters r_o and t.

Along with definition (3-17), we also introduce the following quantities

$$D = \frac{-2rp_o'}{B^2\sigma^2} , \qquad \beta = \frac{2p_o}{B^2} ,$$

$$S = \frac{4B_\theta^2}{B^2\sigma^2} , \qquad R = \frac{rB_\theta}{\eta \sqrt{\rho_o}} ,$$

$$\Theta = \frac{\gamma}{\gamma-1} \cdot \frac{-2r\rho_o}{B^2\sigma^2} T_o' , \qquad M = \frac{\mu_\perp}{rB_\theta \sqrt{\rho_o}} \tag{3-19}$$

$$K_{||} = \frac{\gamma-1}{\gamma} \frac{\kappa_{||}|\sigma|B_\theta}{rB^2 \sqrt{\rho_o}} , \qquad K_\perp = \frac{\gamma-1}{\gamma} \frac{\kappa_\perp}{rB_\theta \sqrt{\rho_o}} ,$$

where every quantity is evaluated at $r = r_o$ as before. The linear stability criterion for the ideal modes is given by the inequality $D < 1/4$, which is known as Suydam's criterion.[13] We assume that this stability condition is always satisfied so that the plasma is ideally stable and the instability arises from the finite diffusion coefficients.

The dependent variables are defined in a dimensionless form as

$$\bar{A} = \frac{1}{r_o B_\theta B}\left(\frac{\sigma}{|\sigma|}\right)A\,, \qquad \bar{\phi} = \frac{\sqrt{\rho_o}}{r_o B_\theta B}\left(\frac{\sigma}{|\sigma|}\right)\phi$$

$$\bar{p} = \frac{2}{B^2|\sigma|^{3/2}}\,p_1\,, \qquad \bar{\nu} = \frac{2\sqrt{\rho_o}}{B^2|\sigma|^{3/2}}\,\nu \tag{3-20}$$

$$\bar{T} = \frac{\gamma}{\gamma-1}\cdot\frac{2p_o}{B^2|\sigma|^{3/2}}\,T_1\,, \qquad \bar{s} = \frac{2p_o}{B^2|\sigma|^{3/2}}\,s_1$$

where we have the relation

$$\bar{s} = \bar{T} - \bar{p} \tag{3-21}$$

from Eq. (3-15). Here we note that $\bar{A}$ and $\bar{\phi}$ are of order δ^2 and $\bar{p}$, $\bar{\nu}$, $\bar{T}$ and $\bar{s}$ are of order δ.

We now rewrite Eqs. (3-7)-(3-10), (3-12) and (3-13) in terms of the new variables and parameters defined above:

$$\frac{d\bar{A}}{d\tau} = \frac{\partial\bar{\phi}}{\partial\bar{\theta}} + \frac{1}{R}\,\bar{\Delta}_\perp\bar{A} \tag{R-1}$$

$$\frac{d}{d\tau}\tilde{\Delta}_{\perp}\tilde{\phi} = \frac{\partial}{\partial\tilde{\theta}}\tilde{\Delta}_{\perp}\tilde{A} + \{\tilde{A},\tilde{\Delta}_{\perp}\tilde{A}\} - \frac{\partial\tilde{p}}{\partial y} + M\tilde{\Delta}^{2}_{\perp}\tilde{\phi} \tag{R-2}$$

$$\frac{d}{d\tau}\tilde{p} = \frac{\gamma\beta}{2+\gamma\beta}\left[S\frac{\partial\tilde{\phi}}{\partial y} + \frac{\partial\tilde{\nu}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{\nu}\} + \frac{1}{R}\tilde{\Delta}_{\perp}\tilde{p}\right]$$

$$- D\frac{\partial\tilde{\phi}}{\partial y} + \frac{2(\gamma-1)}{2+\gamma\beta}\left(K_{||}\tilde{\Delta}_{||} + K_{\perp}\tilde{\Delta}_{\perp}\right)\tilde{T} \tag{R-3}$$

$$\frac{d}{d\tau}\tilde{T} = \frac{\gamma\beta}{2+\gamma\beta}\left[S\frac{\partial\tilde{\phi}}{\partial y} + \frac{\partial\tilde{\nu}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{\nu}\} + \frac{1}{R}\tilde{\Delta}_{\perp}\tilde{p}\right]$$

$$- \Theta\frac{\partial\tilde{\phi}}{\partial y} + \gamma\frac{2+\beta}{2+\gamma\beta}\left(K_{||}\tilde{\Delta}_{||} + K_{\perp}\tilde{\Delta}_{\perp}\right)\tilde{T} \tag{R-4}$$

$$\frac{d\tilde{\nu}}{d\tau} = \frac{\partial\tilde{p}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{p}\} + D\frac{\partial\tilde{A}}{\partial y} + M\tilde{\Delta}_{\perp}\tilde{\nu}\ . \tag{R-5}$$

Either Eqs. (R-3) or (R-4) may be replaced by the entropy equation

$$\frac{d}{d\tau}\tilde{s} = (D-\Theta)\frac{\partial\tilde{\phi}}{\partial y} + \left(K_{||}\tilde{\Delta}_{||} + K_{\perp}\tilde{\Delta}_{\perp}\right)\tilde{T}\ . \tag{R-6}$$

Here we have used the following definitions:

$$\{f,g\} = (\nabla_\perp g \times \nabla_\perp f)\cdot\tilde{b} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{\phi, f\}$$

$$\tilde{\Delta}_{||} f = \frac{\partial^2 f}{\partial\tilde{\theta}^2} + \left(\frac{\sigma}{|\sigma|}\right)\left(\{\tilde{A}, \frac{\partial f}{\partial\tilde{\theta}}\} + \frac{\partial}{\partial\tilde{\theta}}\{\tilde{A}, f\}\right) + \{\tilde{A}, \{\tilde{A}, f\}\} \tag{3-22}$$

$$\tilde{\Delta}_\perp = \left(\frac{\partial}{\partial x} - \tilde{\theta}\frac{\partial}{\partial y}\right)^2 + \frac{\partial^2}{\partial y^2}$$

The boundary conditions of this set of equations will be discussed in the next section.

In the reduced set of equations (R), there are eight parameters, i.e., D, Θ, S, β, R, M, $K_\perp$ and $K_{||}$, which determine the solutions. However, if we apply the following scale transformations, the coefficients 1/R, M and $K_\perp$ in those equations will be formally replaced by 1, RM and $RK_\perp$, respectively:

$$\begin{aligned} &x \to R^{-1/2}\, x\,, && y \to R^{-1/2}\, y \\ &\tilde{A} \to R^{-1}\tilde{A}\,, && \tilde{\phi} \to R^{-1}\tilde{\phi} \\ &\tilde{p} \to R^{-1/2}\,\tilde{p}\,, && \tilde{T} \to R^{-1/2}\,\tilde{T} \\ &\tilde{\nu} \to R^{-1/2}\,\tilde{\nu}\,, && \tilde{s} \to R^{-1/2}\,\tilde{s} \end{aligned} \tag{3-23}$$

Therefore, using the following definition of the new parameters:

$$M_R = MR = \mu_\perp / n\rho_o$$
$$K_R = \gamma K_\perp R = (\gamma-1)\kappa_\perp / n\rho_o \;, \tag{3-24}$$

the solutions of the system transformed from Eqs. (R) by the transformation (3-23) depend only on M_R and K_R as well as D, Θ, S, β, and $K_{||}$, but not on the magnetic Reynolds number R, so long as the boundary conditions do not depend on R after this scale transformation. This is certainly the case if we consider the solutions that decay rapidly as $|x|, |y| \to \infty$.

The set of equations (R) is suitable to describe the resistive g-mode fluctuations in a screw pinch or an RFP, in which the azimuthal (or poloidal) field B_θ and the longitudinal (or toroidal) field B_z are of the same order of magnitude, and the plasma beta β, which is defined as the ratio of the thermal pressure p_o to the magnetic pressure $B_o^2/2$, is of order 1. However, if we consider a straight cylindrical plasma with low β and a strong longitudinal field B_z, we may further simplify the set of equations (R). This further simplified system is a reasonable model of a tokamak plasma with a large aspect ratio.

Let ε_a be the inverse aspect ratio, i.e., $\varepsilon_a = a/L$, where a and L denote the radius and the length of the cylinder, respectively. We assume that ε_a is a small number and that $\beta \sim B_\theta/B_z \sim \varepsilon_a$ and $\sigma \sim \sqrt{\varepsilon_a}$, namely, low β, strong longitudinal field and low shear. The diffusion coefficients such as the resistivity η, the perpendicular viscosity ν and the perpendicular heat conductivity $\kappa_\perp$ are also taken to be of order $\varepsilon_a \cdot \delta^2$. Under these assumptions, it follows that $D \sim \Theta \sim R \sim M \sim K_\perp$ are of order 1 and S is of order ε. It also follows

that the safety factor $q(r)$, defined by $q(r) = rB_z(r)/LB_\theta(r)$, is of order 1, which is the case in tokamaks. By rewriting y as $y = |\sigma|^{1/2}(\sigma/|\sigma|)(z/L - q(r)\theta)/q$, where $\sigma = r_o q'/q$, and z/L corresponds to the toroidal angle, quantities of order ϵ_a^{-1} such as L and μ do not appear explicitly in the equations.

In order to focus on the effect of the temperature gradient, we further assume that the density gradient is small, i.e., $r_o\rho_o'/\rho_o = O(\epsilon_a)$. This assumption makes Eqs. (R-3) and (R-4) identical to each other to the lowest order. In fact, the parameters D and Θ defined in Eqs. (3-19) are now related to each other as

$$D = \frac{\gamma-1}{\gamma}\,\Theta + \frac{\beta}{2}\,N\ , \tag{3-25}$$

where

$$N = \frac{-2r_o\rho_o'}{\rho_o\sigma^2}\ ,$$

is the parameter related to the mean density gradient. Therefore, it follows that $D = (\gamma-1)\Theta/\gamma$. Consequently, the perturbed temperature $\tilde{T}$ and the perturbed entropy $\tilde{s}$ are expressed in $\tilde{p}$ as

$$\begin{aligned} \tilde{T} &= \gamma\tilde{p}/(\gamma-1) \\ \tilde{s} &= \tilde{p}/(\gamma-1)\ . \end{aligned} \tag{3-26}$$

Thus the subsidiary expansion of Eqs. (R-1)-(R-5) in ϵ_a leads to the following set of equations:

$$\frac{d\tilde{A}}{d\tau} = \frac{\partial\tilde{\phi}}{\partial\tilde{\theta}} + \frac{1}{R}\tilde{\Delta}_{\perp}\tilde{A} \tag{T-1}$$

$$\frac{d}{d\tau}\tilde{\Delta}_{\perp}\tilde{\phi} = \frac{\partial}{\partial\tilde{\theta}}\tilde{\Delta}_{\perp}\tilde{A} + \{\tilde{A},\tilde{\Delta}_{\perp}\tilde{A}\} - \frac{\partial\tilde{p}}{\partial y} + M\tilde{\Delta}_{\perp}^{2}\tilde{\phi} \tag{T-2}$$

$$\frac{d}{d\tau}\tilde{p} = -D\frac{\partial\tilde{\phi}}{\partial y} + \chi\tilde{\Delta}_{\perp}\tilde{p} \ . \tag{T-3}$$

Here $\chi = YK_{\perp}$ and the equation for $\tilde{\nu}$ is given by (R-5), which is, however, decoupled from the set of equations (T). Although we ignored the parallel heat conductivity in the system (T), we can introduce it simply by replacing $\chi\tilde{\Delta}_{\perp}$ by $\chi_{||}\Delta_{||} + \chi\tilde{\Delta}_{\perp}$ in Eq. (T-3), where $\chi_{||} = YK_{||}$. The boundary conditions to be imposed on this system will be discussed in Sections IV and V. We note that, as in the case of the system (R), the scale transformations (3-23) replace the coefficients $1/R$, M and χ in Eqs. (T) by 1, M_R and K_R, respectively, so that the solutins of the transformed system depend only on the parameters D, M_R and K_R, if we impose proper boundary conditions.

These two reduced sets of equations, i.e., Eqs. (R) and Eqs. (T), form the bases of our models of the nonlinear resistive pressure driven mode. From the next section on, we will investigate these reduced equations in order to characterize the anomalous transport caused by these fluctuations.

IV. COUPLING OF THE DYNAMO EFFECT WITH THE ANOMALOUS HEAT TRANSPORT IN RFP PLASMAS

In this section, we will consider the initial-boundary value problem of the system of Eqs. (R) and derive a relationship between the dynamo effect, or the anomalous electric field $\underset{\sim}{\varepsilon}$ defined in Eq. (2-9), and the anomalous heat transport $\langle s_1 \underset{\sim}{v}_{1r} \rangle$: in fact, we will show that the relationship given by Eq. (2-13) holds locally, not as the integral over the total plasma volume, if we consider the fluctuations caused by the resistive g-mode fluctuations. Before deriving this relationship, however, we will first discuss the boundary conditions of the system of Eqs. (R) and subsequently, the averaging operations applying to the solutions of this system.

In order to make it easy to compare the solutions of the system to the more familiar linear solutions,[8] we transform[14] the independent variables, x, y and $\bar{\theta}$ to the new variables $\bar{x} = x$, $\bar{y} = y + \bar{\theta}x$ and $\bar{z} = \bar{\theta}$. This transformation transforms the derivatives in Eqs. (R) as follows: $\partial/\partial x = \partial/\partial\bar{x} + \bar{z}\partial/\partial\bar{y}$, $\partial/\partial y = \partial/\partial\bar{y}$, $\partial/\partial\bar{\theta} = \partial/\partial\bar{z} + \bar{x}\partial/\partial\bar{y}$ and $\bar{\Delta}_\perp = \partial^2/\partial^2\bar{x} + \partial^2/\partial^2\bar{y}$. Although we will generally consider the solutions which depend on $\bar{z}$ as well as on $\bar{x}$ and $\bar{y}$, the well known linear solutions of the resistive fast interchange mode[8] is the $\bar{z}$-independent solution of the linearized version of the system of equations (R).

Throughout this thesis, we will use two coordinate systems, $(x,y,\bar{\theta})$ and $(\bar{x},\bar{y},\bar{z})$, using whichever system makes equations simpler. However, it is easier to write the boundary conditions of the system in terms of $(\bar{x},\bar{y},\bar{z})$ rather than $(x,y,\bar{\theta})$ because of the periodicity of the $\bar{z}$-coordinate. From the definition, the $\bar{z}$-direction with fixed $\bar{x}$ and $\bar{y}$ is the direction of the mean magnetic field $\bar{B}_0$ at $r = r_0$. Since the

surface $r = r_0$ is taken to be a rational surface, every field line on this surface is closed. Therefore the particular mean field line we consider on this surface comes back to its original position after a finite length $\tilde{z} = L_z$. Thus, provided that this particular mean field line never gets too close to itself on this surface before coming back to its original position, it is natural to impose the periodic boundary condition in $\tilde{z}$ with period L_z, together with proper boundary conditions in $\tilde{x}$ and $\tilde{y}$. On the other hand, the $\tilde{\theta}$-direction with fixed x and y is the direction of the mean field line starting from a point off the rational magnetic surface (if $x \neq 0$) so that there is no periodicity in $\tilde{\theta}$ generally.

Thus we impose the following boundary conditions which are similar to those imposed on the linearized problem of the system (R):

(I) $\tilde{\phi},\ \tilde{\Delta}_\perp\tilde{\phi},\ \tilde{A},\ \tilde{p},\ \tilde{\nu},\ \tilde{T} = 0$ at $|\tilde{x}| = \delta_x$

(II) $\tilde{\phi},\ \tilde{\Delta}_\perp\tilde{\phi},\ \tilde{A},\ \tilde{p},\ \tilde{\nu},\ \tilde{T} = 0$ at $|\tilde{y}| = \delta_y$

or, instead of (II),

(II') $\tilde{\phi},\ \tilde{\Delta}_\perp\tilde{\phi},\ \tilde{A},\ \tilde{p},\ \tilde{\nu}$, and $\tilde{T}$ and their $\tilde{y}$-derivatives are periodic with period $2\delta_y$ in $\tilde{y}$

and

(III) $\tilde{\phi},\ \tilde{\Delta}_\perp\tilde{\phi},\ \tilde{A},\ \tilde{p},\ \tilde{\nu}$, and $\tilde{T}$ are periodic with period L_z in the $\tilde{z}$-direction.

Here δ_x, δ_y and L_z are the prescribed positive values such that $\delta << \delta_x, \delta_y << a$ and $L_z/a \geq O(1)$, where δ is the small scale parameter introduced in Section III and a is the plasma minor radius. δ_x and δ_y give the size of the boundary layer whereas δ give the size of the resistive layer. In the linear theory in Ref. 8, δ_x/δ is taken to be ∞, in which case the asymptotic behavior of the linear solution at large $|\tilde{x}|$ (i.e., $|\tilde{x}| >> \delta$) takes an exponentially decaying form. We presume from this linear asymptotic analysis that the solutions of the nonlinear system (R) are essentially independent of the choice of δ_x and decay rapidly as $|\tilde{x}|$ becomes large, as long as δ_x/δ is taken to be large enough. Appendix B contains further discussions of the boundary conditions.

It is natural to define averaging operations in the following way for a function $f(\tilde{x},\tilde{y},\tilde{z},\tau;r_o,t)$ satisfying boundary conditions (I), (II) [or (II')] and (III):

$$\langle f\rangle = \frac{1}{4\delta_x\delta_y L_z}\int_{-\delta_x}^{\delta_x} d\tilde{x}\int_{-\delta_y}^{\delta_y} d\tilde{y}\int_0^{L_z} d\tilde{z}\; f(\tilde{x},\tilde{y},\tilde{z},\tau;r_o,t) \qquad (4\text{-}1)$$

$$\overline{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T d\tau f(\tilde{x},\tilde{y},\tilde{z},\tau;r_o,t)\ . \qquad (4\text{-}2)$$

It is clear from the boundary conditions that the $\langle\ \rangle$ average of any spatial derivative of f with respect to $\tilde{x}$, $\tilde{y}$ or $\tilde{z}$ vanishes. We also define the averaging $\overline{\langle\ \rangle}$ as the combination of these two averaging, i.e., $\overline{\langle f\rangle} \equiv \overline{\langle f\rangle}$. It is easy to check that $\langle\ \rangle$, $\overline{\ }$ and $\overline{\langle\ \rangle}$ satisfy the Reynolds conditions (2-5), where the condition (2-5d) holds only in terms of the slowly varying variables r_o and t.

We point out that if we take δ_x to be large enough and we obtain

the solution f of the system (R) that decays rapidly as $|\tilde{x}|$ becomes large, then the average $\langle f \rangle$ of the solution f given by Eq. (4-1) is not well defined. In fact, provided that such solutions f depend only weakly on the choice of δ_x for large enough δ_x, the integral in $\tilde{x}$ used in Eq. (4-1)

$$\int_{-\delta_x}^{\delta_x} f \, d\tilde{x} \rightarrow \int_{-\infty}^{\infty} f \, d\tilde{x} < \infty$$

as $\delta_x \rightarrow \infty$, and so the value $\langle f \rangle$ scales as $1/\delta_x$ for a large δ_x. In order to keep the physical meaning of the average of this kind of solution, we redefine the operator < > as

$$\langle f \rangle = \frac{1}{2\delta_y \Delta L_z} \int_{-\infty}^{\infty} d\tilde{x} \int_{-\delta_y}^{\delta_y} d\tilde{y} \int_0^{L_z} d\tilde{z} \; f(\tilde{x},\tilde{y},\tilde{z};r_o,t) \; . \qquad (4\text{-}3)$$

If we choose the boundary condition (II), instead of (II'), then we also need to modify the averaging in the $\tilde{y}$-direction in a similar way. Here we note the < > averaging given by Eq. (4-3) does not satisfy the condition (2-5) of the Reynolds conditions, and therefore it is not an averge in a rigorous sense. However, this modified average < > provides the contribution of the fluctuating quantity per unit volume.

In terms of the scaled variables defined in Eqs. (3-16) and (3-20) and the averaging operator < > defined above [either Eq. (4-1) or Eq. (4-3)], the anomalous heat flux $-p_o\langle s_1 v_{1r}\rangle$, where the radial velocity $v_{1r} = \underset{\sim}{v}\cdot\nabla r$, is expressed as

$$-p_o\langle s_1 v_{1r}\rangle = \frac{B_\theta B^2\sigma^2}{\sqrt{\rho_o}}\langle\tilde{s}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle . \qquad (4\text{-}4)$$

On the other hand, the averaged electric field $\langle \underset{\sim}{v}_1 \times \underset{\sim}{b}_1\rangle$ can be written as

$$\langle \underset{\sim}{v}_1\times\underset{\sim}{b}_1\rangle = \langle(\nabla_\perp\phi\times\nabla_\perp A)\cdot\underset{\sim}{b}\rangle\underset{\sim}{b} + \langle\frac{p_1}{B^2}\nabla_\perp\phi - \frac{\nu}{B^2}\nabla_\perp A\rangle$$

$$= \frac{\sigma B_\theta^2 B}{\sqrt{\rho_o}}\langle\frac{\partial\tilde{A}}{\partial x}\frac{\partial\tilde{\phi}}{\partial y} - \frac{\partial\tilde{A}}{\partial y}\frac{\partial\tilde{\phi}}{\partial x}\rangle\underset{\sim}{b} + \frac{B_\theta B\sigma|\sigma|}{2\sqrt{\rho_o}}\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial x} - \tilde{\nu}\,\frac{\partial\tilde{A}}{\partial x}\rangle\nabla r$$

$$+ \frac{B_\theta\sigma^2}{2\sqrt{\rho_o}}\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y} - \tilde{\nu}\,\frac{\partial\tilde{A}}{\partial y}\rangle\,(\nabla r\times\underset{\sim}{B}_o) ,$$

to the lowest order. In particular, for $\underset{\sim}{\varepsilon} = \langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle - \frac{1}{\rho_o}\langle\rho_1\underset{\sim}{v}_1\rangle\times\underset{\sim}{B}_o$, it follows

$$\underset{\sim}{\varepsilon}\cdot\underset{\sim}{B}_o = \frac{\sigma B_\theta^2 B}{\sqrt{\rho_o}}\langle\{\tilde{A},\tilde{\phi}\}\rangle$$

and

$$\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o = \frac{p_o' B_\theta\sigma^2}{2\sqrt{\rho_o}}\,\frac{2+\gamma\beta}{\gamma\beta}\left(-\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle + \frac{2(\gamma-1)}{2+\gamma\beta}\langle\tilde{s}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle + \frac{\gamma\beta}{2+\gamma\beta}\langle\tilde{\nu}\,\frac{\partial\tilde{A}}{\partial y}\rangle\right) . \qquad (4\text{-}5)$$

Here we have used the relations

$$\underset{\sim}{J}_o = \left(\frac{\underset{\sim}{J}_o \cdot \underset{\sim}{B}_o}{B^2}\right)\underset{\sim}{B}_o + \frac{\underset{\sim}{B}_o \times \nabla p_o}{B^2}$$

$$s_1 = \frac{1}{\gamma-1}\frac{p_1}{p_o} - \frac{\gamma}{\gamma-1}\frac{\rho_1}{\rho_o}$$

We also note that $\{\tilde{A},\tilde{\phi}\}$ has a divergence form, i.e.,

$$\{\tilde{A},\tilde{\phi}\} = \frac{\partial}{\partial x}\left(-\tilde{\phi}\,\frac{\partial\tilde{A}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\tilde{\phi}\,\frac{\partial\tilde{A}}{\partial x}\right) ,$$

so $\langle\{\tilde{A},\tilde{\phi}\}\rangle$ vanishes because of the boundary conditions. Therefore $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{B}_o$ is of order δ^3 while $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o$ is of order δ^2 for the resistive fast interchange mode.

We now confirm Eq. (2-11), which relates the dynamo term $\underset{\sim}{\varepsilon}$ and the anomalous heat flux $\langle s_1 v_{1r}\rangle$, directly from the set of the reduced equations (R). Adding up $\tilde{\Delta}_\perp\tilde{A}$ × Eq. (R-1) and $\tilde{\phi}$ × Eq. (R-2) and taking the < > average of the resulting equation yields an equation for the energy of the perpendicular magnetic field and the perpendicular flow

$$\frac{\partial}{\partial\tau}\frac{1}{2}\left(\langle|\tilde{\nabla}_\perp\tilde{A}|^2 + |\tilde{\nabla}_\perp\tilde{\phi}|\rangle\right) = -\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle - \frac{1}{R}\langle|\tilde{\Delta}_\perp\tilde{A}|^2\rangle - M\,\langle|\tilde{\Delta}_\perp\tilde{\phi}|^2\rangle , \quad (4\text{-}6)$$

where

$$\tilde{\nabla}_\perp = \nabla r\left(\frac{\partial}{\partial x} - \tilde{\theta}\,\frac{\partial}{\partial y}\right) + \frac{\nabla r\times\underset{\sim}{B}_o}{|\underset{\sim}{B}_o|}\,\frac{\partial}{\partial y} .$$

Likewise, by adding up Eqs. (R-3), (R-5) and (R-6) with the following factors:

$$\tilde{p}\cdot(\text{R-3}) + \frac{\gamma\beta}{2+\gamma\beta}\,\tilde{v}\cdot(\text{R-5}) + \frac{2(\gamma-1)}{2+\gamma\beta}\,\tilde{s}\cdot(\text{R-6})$$

and by taking the < > average, we obtain

$$\frac{1}{2}\frac{\partial}{\partial\tau}\left(\langle\tilde{p}^2\rangle + \frac{\gamma\beta}{2+\gamma\beta}\langle\tilde{v}^2\rangle + \frac{2(\gamma-1)}{2+\gamma\beta}\langle\tilde{s}^2\rangle\right)$$

$$= -\left(D - \frac{\gamma\beta}{2+\gamma\beta}S\right)\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle - \frac{2(\gamma-1)}{2+\gamma\beta}(\Theta-D)\,\langle\tilde{s}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle$$

$$+ \frac{\gamma\beta}{2+\gamma\beta}D\langle\tilde{v}\,\frac{\partial\tilde{A}}{\partial y}\rangle - \frac{\gamma\beta}{2+\gamma\beta}\frac{1}{R}\langle|\tilde{\nabla}_\perp\tilde{p}|^2\rangle - \frac{\gamma\beta}{2+\gamma\beta}M\langle|\nabla_\perp\tilde{v}|^2\rangle$$

$$- \frac{2(\gamma-1)}{2+\gamma\beta}\left(K_{||}\langle\left|\frac{\partial\tilde{T}}{\partial\tilde{\theta}} - \left(\frac{\sigma}{|\sigma|}\right)\{\tilde{A},\tilde{T}\}\right|^2\rangle + K_\perp\langle|\tilde{\nabla}_\perp\tilde{T}|^2\rangle\right)\,. \qquad (4\text{-}7)$$

For the solutions such as saturated modes or stationary turbulence, we can drop the left-hand-sides of Eqs. (4-6) and (4-7) by taking the long time average in τ. Taking the - average of Eqs. (4-6) and (4-7) and rearranging the terms, we obtain

$$\overline{\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle} = -\frac{1}{R}\overline{\langle|\tilde{\Delta}_\perp\tilde{A}|^2\rangle} - M\overline{\langle|\tilde{\Delta}_\perp\tilde{\phi}|^2\rangle} \qquad (4\text{-}8)$$

and

$$-D\left(\overline{\langle\tilde{p}}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle - \frac{2(\gamma-1)}{2+\gamma\beta}\,\overline{\langle\tilde{s}}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle - \frac{\gamma\beta}{2+\gamma\beta}\,\overline{\langle\tilde{v}}\,\frac{\partial\tilde{A}}{\partial y}\rangle\right) - \frac{2(\gamma-1)}{2+\gamma\beta}\,\Theta\,\overline{\langle\tilde{s}}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle$$

$$= -\frac{\gamma\beta}{2+\gamma\beta}\,S\overline{\langle\tilde{p}}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle + \frac{\gamma\beta}{2+\gamma\beta}\,\frac{1}{R}\,\overline{\langle|\tilde{\nabla}_{\perp}\tilde{p}|^2\rangle} + \frac{\gamma\beta}{2+\gamma\beta}\,M\,\overline{\langle|\tilde{\nabla}_{\perp}\tilde{v}|^2\rangle}$$

$$+ \frac{2(\gamma-1)}{2+\gamma\beta}\left(K_{||}\overline{\langle\left|\frac{\partial\tilde{T}}{\partial\tilde{\theta}} - \left(\frac{\sigma}{|\sigma|}\right)\{\tilde{A},\tilde{T}\}\right|^2\rangle} + K_{\perp}\overline{\langle|\nabla_{\perp}\tilde{T}|^2\rangle}\right) . \qquad (4\text{-}9)$$

It is clear that the fourth term of the left-hand side of Eq. (4-9) is proportional to $\langle s_1 v_{1r}\rangle$ and the sum of the first three terms is proportional to $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o$ from Eqs. (4-4) and (4-5). By substituting Eq. (4-8) into $\overline{\langle\tilde{p}}\,\partial\tilde{\phi}/\partial y\rangle$ in the right-hand-side of Eq. (4-9), we see that the right-hand-side is a non-negative function related to the dissipation of the system. In fact, multiplying Eq. (4-9) by $-(B^2B_\theta\sigma^4)(2+\gamma\beta)/4\gamma\beta r_o\sqrt{\rho_o}$ yields the following relation:

$$\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o + \overline{\langle s_1 v_{1r}\rangle}\rho_o T_o'$$

$$= -\eta\,\overline{\langle|\underset{\sim}{J}_1|\rangle^2} - \mu_{\perp}\,\overline{\langle|\underset{\sim}{w}_1|\rangle^2} - \frac{1}{T_o}\left(\kappa_{||}\overline{\langle|\nabla_{||}T_1|^2\rangle} + \kappa_{\perp}\overline{\langle|\nabla_{\perp}T_1|^2\rangle}\right) . \qquad (4\text{-}10)$$

Here we have used

$$|\underset{\sim}{J}_1|^2 = \frac{B^2\sigma^4}{4r_o}\left(S|\tilde{\Delta}_{\perp}\tilde{A}|^2 + |\tilde{\nabla}_{\perp}\tilde{p}|^2\right)$$

and

$$|\underset{\sim}{w}_1|^2 = \frac{B^2\sigma^4}{4r_o\rho_o}\left(S|\tilde{\Delta}_{\perp}\tilde{\phi}|^2 + |\tilde{\nabla}_{\perp}\tilde{v}|^2\right) ,$$

which are derived from the following relations:

$$\underset{\sim}{J}_1 = \nabla\times\underset{\sim}{B}_1 = -(\Delta_\perp A)\underset{\sim}{b} - \nabla_\perp p_1\times\underset{\sim}{b}$$

$$\underset{\sim}{w}_1 = \nabla\times\underset{\sim}{v}_1 = -(\Delta_\perp \phi)\underset{\sim}{b} - \nabla_\perp \nu\times\underset{\sim}{b}\ .$$

We have thus shown that Eq. (2-13) holds locally, in other words, Eq. (2-11) holds on each rational surface with $f = 0$.

To conclude, we have derived in Eq. (4-10) a relation between the dynamo effect and the anomalous heat transport on each rational surface of an RFP plasma, based on the system (R) of the resistive g-mode fluctuations. This strong coupling of the dynamo effect with the anomalous heat transport explains the large heat loss or the short energy confinement time of RFP plasmas, where it is believed that the dynamo plays an important role in sustaining the equilibrium configuration through relaxation processes.

V. SOME PROPERTIES OF THE ANOMALOUS HEAT TRANSPORT OF TOKAMAK PLASMAS

We now turn to a model of the resistive g-mode fluctuations in a tokamak described by the system of equations (T). In this section, we will show that the dynamo effect decouples from the anomalous heat transport in tokamak plasmas and, therefore, the anomalous heat flux transports heat in the direction opposite to the mean temperature gradient.

As in the RFP case, the initial-boundary value problem of this sytem is posed more easily by introducing the transformation of the independent variables $\tilde{x} = x$, $\tilde{y} = y + \tilde{\theta}x$ and $\tilde{z} = \tilde{\theta}$. The boundary conditions for $\tilde{A}$, $\tilde{\phi}$, $\Delta\tilde{\phi}$ and $\tilde{p}$ in terms of the new independent variables $\tilde{x},\tilde{y},\tilde{z}$ are also given by (I), (II) [or (II')] and (III) in the previous section. (See also Appendix B.) The averaging operators < > and $\bar{\ }$ defined in the previous section are also used in this section.

The relation between the dynamo dissipation $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o$ and the anomalous heat transport $-p_o\langle s_1 \underset{\sim}{v}\rangle$ given by Eq. (4-10) for an RFP plasma is considerably simplified for a tokamak plasma. Since $\tilde{s} = \tilde{p}/(\gamma-1)$ [Eq. (3-26)] for the tokamak case, Eq. (4-5) becomes

$$\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o = \frac{p_o' B_\theta \sigma^2}{2\sqrt{\rho_o}} \langle \tilde{v} \frac{\partial \tilde{A}}{\partial y}\rangle \tag{5-1}$$

to the lowest order under the expansion in the inverse aspect ratio ε_a. From the coefficients of the right-hand-side of Eq. (5-1), it is clear that $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o$ is of order ε_a^3. Here we recall that we measure the length in r_o units and the magnetic field in B units, so that from the assumption

of the ε_a-expansion discussed in Section III, we have $p_o' \sim B_\theta \sim \sigma^2 \sim \varepsilon_a$. On the other hand, the radial component of the anomalous heat flux $-p_o\langle s_1 v_{1r}\rangle$ given by Eq. (4-4) is of order ε_a^2, which is of the same order as the collisional heat transport $\kappa_\perp T_o'$. Thus the heat production $\underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o$ related to the dynamo effect is smaller by an order of the inverse aspect ratio ε_a than the anomalous heat transport.

Taking into account the ordering in ε_a and recalling the assumption that $K_{||}$ does not enter the equations, we can simplify Eq. (4-9) as

$$-D\overline{\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle} = \chi\overline{\langle|\tilde{\nabla}_\perp\tilde{p}|^2\rangle} \ . \qquad (5\text{-}2)$$

Here Eq. (3-26), and the relations $\Theta = \gamma D/(\gamma-1)$ and $\gamma K_\perp = \chi$ are used: we note that the anomalous heat flux $-p_o\langle s_1 v_{1r}\rangle$ is proportional to $\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle$ in the tokamak case. We may also obtain Eq. (5-2) in the following way: by multiplying Eq. (T-3) by $\tilde{p}$ and taking its < > average, we get

$$\frac{1}{2}\frac{\partial}{\partial\tau}\langle|\tilde{p}|^2\rangle = -D\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle - \chi\langle|\tilde{\nabla}_\perp\tilde{p}|^2\rangle \ . \qquad (5\text{-}3)$$

Taking the $\bar{}$ average of this equation yields Eq. (5-2).

In terms of the physical variables, Eq. (5-2) may be written as

$$-\rho_o T_o' \overline{<} s_1 v_{1r} > = \frac{\kappa_\perp}{T_o} \overline{<} |\nabla_\perp T_1|^2 > , \tag{5-4}$$

which corresponds to Eq. (4-10), where we recall our assumption that $p_o' = \rho_o T_o'$ for the system (T). Here we used Eqs. (3-16), (3-20) and (3-26).

To conclude, we have shown that the anomalous heat transport is decoupled from the dynamo effect in tokamak plasmas. We have also shown from Eq. (5-4) that the anomalous heat flux across the magnetic surface is in the direction opposite to the mean temperature gradient.

VI. NONLINEAR STABILITY ANALYSIS OF A TOKAMAK PLASMA

In this section, we will discuss the stability of the null solution of Eqs. (T). The boundary conditions to be used in this section are (I), (II) [or (II')] and (III) given in Section IV and V. The null solution of this system represents the equilibrium state of the plasma. We define the stability of the null solution in the following way: let $\varepsilon(\tau)$ be the energy norm of the system defined as

$$\varepsilon(\tau) = \frac{1}{2} \langle |\tilde{\nabla}_{\perp} \tilde{A}|^2 + |\tilde{\nabla}_{\perp} \tilde{\phi}|^2 + |\tilde{p}|^2 \rangle \ .$$

Although this $\varepsilon(\tau)$ is not the total energy of the perturbed plasma, we take $\varepsilon(\tau)$ to be a measure of the amplitude of those perturbed quantities. The null solution of the system (T) is called asymptotically stable (in the sense of Liapunov) if there is a positive number δ_I such that for any initial condition the energy norm of which is less than δ_I, i.e., for $\varepsilon(0) < \delta_I$, the energy norm approaches zero for large τ, i.e., $\varepsilon(\tau) \to 0$ as $\tau \to \infty$. In particular, if δ_I is finite, it is called conditionally stable and if $\delta_I = \infty$, then it is called globally stable. The globally stable null solution is also called monotonically stable if $d\varepsilon(\tau)/d\tau < 0$ for all $\tau > 0$.

The various notions of stability defined above are based on the nonlinearity of the system. For a perturbation with infinitesimal amplitude, however, we have the spectral problem of the linearized system of Eqs. (T). Assuming that the solutions of the linearized system depend exponentially on τ as $e^{q\tau}$, where q is a complex number, we replace $\partial/\partial\tau$ by q. Provided there exist such numbers q for which the linearized system has nontrivial solutions, the numbers q are

called the eigenvalues of the system. The linear stability of the null solution is then defined in connection with the eigenvalues q as follows: the null solution is called linearly stable if there are no eigenvalues such that $\mathrm{Re}\, q > 0$; marginally stable if there is at least one eigenvalue with $\mathrm{Re}\, q = 0$ and all the other eigenvalues have $\mathrm{Re}\, q \leq 0$; and unstable if at least one eigenvalue has $\mathrm{Re}\, q > 0$.

In Section VII, we will discuss the case where the null solution of the system (T) is linearly unstable. A small perturbation given to such a system initially grows with a growth rate q and, possibly, saturates eventually with finite amplitude. In order to look for such solutions of the system, it is useful to know when the system becomes linearly unstable. The following proposition asserts that $D \leq 0$ is a sufficient condition for linear stability. In other words, the system may be linearly unstable only when $D > 0$.

<u>PROPOSITION 1</u>: Suppose R, M and χ are all positive constants. If $D \leq 0$, then $\mathrm{Re}\, q < 0$.

<u>PROOF</u>: The linearized equations of the system (T) are

$$q\tilde{A} = \frac{\partial\tilde{\phi}}{\partial\tilde{\theta}} + \frac{1}{R}\tilde{\Delta}_{\perp}\tilde{A} \tag{6-1a}$$

$$q\tilde{\Delta}_{\perp}\tilde{\phi} = \frac{\partial}{\partial\tilde{\theta}}\tilde{\Delta}_{\perp}\tilde{A} - \frac{\partial\tilde{p}}{\partial y} + M\tilde{\Delta}_{\perp}^{2}\tilde{\phi} \tag{6-1b}$$

$$q\tilde{p} = -D\frac{\partial\tilde{\phi}}{\partial y} + \chi\tilde{\Delta}_{\perp}\tilde{p} . \qquad (6\text{-}1c)$$

Here we allow $\tilde{A}$, $\tilde{\phi}$, $\tilde{p}$ and q to take complex values. Adding up $\tilde{\Delta}_{\perp}A$ × Eq. (6-1a)* and $\tilde{\phi}^*$ × Eq. (6-1b), and taking the < > average of the resulting equation, we obtain

$$q^*\langle|\tilde{\nabla}_{\perp}\tilde{A}|^2\rangle + q\langle|\tilde{\nabla}_{\perp}\tilde{\phi}|^2\rangle = -\langle\tilde{p}\frac{\partial\tilde{\phi}^*}{\partial y}\rangle - \frac{1}{R}\langle|\tilde{\Delta}_{\perp}\tilde{A}|^2\rangle - M\langle|\tilde{\Delta}_{\perp}\tilde{\phi}|^2\rangle . \qquad (6\text{-}2)$$

Here * denotes the complex conjugate. Similarly, by multiplying Eq. (6-1c)* by $\tilde{p}$ and taking the < > average, we have

$$q^*\langle|\tilde{p}|^2\rangle = -D\langle\tilde{p}\frac{\partial\tilde{\phi}^*}{\partial y}\rangle - \chi\langle|\tilde{\nabla}_{\perp}\tilde{p}|^2\rangle . \qquad (6\text{-}3)$$

If $D = 0$, then the real part of Eq. (6-3) becomes

$$(\mathrm{Re}\ q)\langle|\tilde{p}|^2\rangle = -\chi\langle|\tilde{\nabla}_{\perp}\tilde{p}|^2\rangle .$$

If $\langle|\tilde{\nabla}_{\perp}\tilde{p}|^2\rangle \neq 0$, then Re $q < 0$. If $\langle|\nabla_{\perp}\tilde{p}|^2\rangle = 0$, or $\tilde{p} \equiv 0$ because of the boundary conditions, then the real part of Eq. (6-2) becomes

$$(\mathrm{Re}\ q)(\langle|\tilde{\nabla}_{\perp}\tilde{A}|^2\rangle + \langle|\tilde{\nabla}_{\perp}\tilde{\phi}|^2\rangle) = -\frac{1}{R}\langle|\tilde{\Delta}_{\perp}\tilde{A}|^2\rangle - M\langle|\tilde{\Delta}_{\perp}\tilde{\phi}|^2\rangle .$$

Therefore Re q is negative for any nontrivial solutions of the system (6-1). If $D < 0$, then eliminating the term $\langle\tilde{p}\partial\tilde{\phi}^*/\partial y\rangle$ from Eqs. (6-2) and (6-3) and taking the real part of the resulting equation yields

$$-(\mathrm{Re}\ q)\left(\langle|\tilde{\nabla}_\perp \tilde{A}|^2\rangle + \langle|\tilde{\nabla}_\perp \tilde{\phi}|^2\rangle - \frac{1}{D}\langle|\tilde{p}|^2\rangle\right)$$

$$= \frac{1}{R}\langle|\tilde{\Delta}_\perp \tilde{A}|^2\rangle + M\langle|\tilde{\Delta}_\perp \tilde{\phi}|^2\rangle - \frac{\chi}{D}\langle|\tilde{\nabla}_\perp \tilde{p}|^2\rangle \qquad (6\text{-}4)$$

Therefore, for the nontrivial solutions of the system (6-1)-(6-3), we have $\mathrm{Re}\ q < 0$.

(Q.E.D.)

We now turn to the nonlinear modes. In previous sections, we discussed some properties of the solutions that satisfy the conditions that the $\bar{\ }$ averages of $\partial\langle|\tilde{\nabla}_\perp \tilde{A}|^2\rangle/\partial\tau$, $\partial\langle|\tilde{\nabla}_\perp \tilde{\phi}|^2\rangle/\partial\tau$ and $\partial\langle|\tilde{p}|^2\rangle/\partial\tau$ vanish. The saturated modes and stationary turbulence, which we are mainly interested in, are the "nontrivial" special cases of solutions with such a property: $\langle|\tilde{\nabla}_\perp \tilde{A}|^2\rangle$ $\langle|\tilde{\nabla}_\perp \tilde{\phi}|^2\rangle$ and $\langle|\tilde{p}|^2\rangle$ are bounded in time τ (so $\varepsilon(\tau)$ is bounded in time τ and $\overline{d\varepsilon(\tau)/d\tau} = 0$), but none of them vanish at $\tau \to +\infty$. However this kind of solutions may not exist for a certain range of parameters of R, M, χ and D. In fact, the following proposition asserts that if the heat conductivity χ is taken to be 0, then there is no such saturated mode nor stationary turbulence. This shows the importance of the heat conductivity χ to sustain the anomalous heat transport for a significantly long time scale.

<u>PROPOSITION 2</u>: Suppose R and M are positive constants, $\chi = 0$ and $D \neq 0$. If $\varepsilon(\tau)$ is bounded in time τ, then $\langle|\tilde{\nabla}_\perp \tilde{A}|^2\rangle$, $\langle|\tilde{\nabla}_\perp \tilde{\phi}|^2\rangle$ and $\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle \to 0$ as $\tau \to \infty$.

<u>PROOF</u>: Before proving this proposition, we recall that $\langle \tilde p \partial\tilde\phi/\partial y\rangle$ is proportional to the anomalous heat flux. Therefore this proposition asserts that the $\bar{\ }$ average of the anomalous heat flux is zero if $\chi = 0$. From now on, we again consider real-valued solutions $\tilde A$, $\tilde\phi$ and $\tilde p$ of the system (T). From the assumption, the $\bar{\ }$ averages of $\partial\langle|\tilde\nabla_\perp \tilde A|^2\rangle/\partial\tau$, $\partial\langle|\tilde\nabla_\perp \tilde\phi|^2\rangle/\partial\tau$ and $\partial\langle|\tilde p|^2\rangle/\partial\tau$ vanish, and, therefore, we have Eqs. (4-8) and (5-2). If $\chi = 0$ then we obtain $\overline{\langle \tilde p \frac{\partial\tilde\phi}{\partial y}\rangle} = \overline{\langle|\tilde\Delta_\perp \tilde A|^2\rangle} = \overline{\langle|\tilde\Delta_\perp \tilde\phi|^2\rangle} = 0$. Since $\langle|\tilde\Delta_\perp \tilde A|^2\rangle$ and $\langle|\tilde\Delta_\perp \tilde\phi|^2\rangle$ are positive functions of τ, therefore, $\langle|\tilde\Delta_\perp \tilde A|^2\rangle$, $\langle|\tilde\Delta_\perp \tilde\phi|^2\rangle \to 0$ as $\tau \to 0$.

In order to prove that $\langle|\tilde\nabla_\perp \tilde A|^2\rangle$ and $\langle|\tilde\nabla_\perp \tilde\phi|^2\rangle \to 0$, we will use the Poincare's inequality:[15] for any real-valued function f (with a proper differentiability condition) satisfying the boundary conditions (I), (II) [or (II')] and (III), there exists a positive constant C such that

$$\langle|\tilde\nabla_\perp f|^2\rangle \le C\langle|\tilde\Delta_\perp f|^2\rangle .$$

Here we note that the constant C is finite since the domain on which f is defined is finite. Applying the Poincare's inequality to $\tilde A$ and $\tilde\phi$ and using the fact that $\langle|\tilde\Delta_\perp \tilde A|^2\rangle$ and $\langle|\tilde\Delta_\perp \tilde\phi|^2\rangle \to 0$ as $\tau \to \infty$, we obtain $\langle|\tilde\nabla_\perp \tilde A|^2\rangle$ and $\langle|\tilde\nabla_\perp \tilde\phi|^2\rangle \to 0$ as $\tau \to +\infty$. To prove that $\langle \tilde p \partial\tilde\phi/\partial y\rangle \to 0$ as $\tau \to 0$, we use Schwartz's inequality:

$$\begin{aligned} |\langle \tilde p \frac{\partial\tilde\phi}{\partial y}\rangle|^2 &\le \langle|\tilde p|^2\rangle \, \langle|\frac{\partial\phi}{\partial y}|^2\rangle \\ &\le \langle|\tilde p|^2\rangle \, \langle|\tilde\nabla_\perp \phi|^2\rangle . \end{aligned}$$

Since $\langle|\tilde{p}|^2\rangle$ is assumed to be bounded in time τ, $\langle\tilde{p}\ \partial\tilde{\phi}/\partial y\rangle \to 0$ as $\tau \to 0$.

(Q.E.D.)

From Proposition 1 and Proposition 2, we find that we need positive R, M, χ and D in order to look for the solutions $\tilde{A}$, $\tilde{\phi}$ and $\tilde{p}$ that grow from small initial values and eventually saturate with finite amplitude or lead to stationary turbulence. In the following theorem, we will show that, if the plasma is sufficiently viscous or heat conductive, the null solution is still stable even for some positive D.

THEOREM 1: Suppose R, M and χ are all positive constants. If either M or χ is large enough, there is a positive critical value D_c of D such that for any $D < D_c$, the null solution is monotonically stable.

PROOF:

Let H be the set of all real valued functions (with sufficient smoothness; for a more specific definition of H, see Appendix C) of $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ satisfying the boundary conditions (I), (II) [or (II')] and (III) given in Section IV. The solutions $\tilde{A}$, $\tilde{\phi}$ and $\tilde{p}$ of the system (T) are obviously some elements of H. For any element f in H, we have the following Poincare's inequalities as before: there exists a constant C such that

$$\langle f^2\rangle \le C\langle|\tilde{\nabla}_\perp f|^2\rangle$$
$$\langle|\tilde{\nabla}_\perp f|^2\rangle \le C\langle|\tilde{\Delta}_\perp f|^2\rangle \ . \qquad (6\text{-}5)$$

By adding Eqs. (4-6) and (5-3), we obtain

$$\frac{d}{d\tau}\,\varepsilon(\tau) = -(1+D)\,\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle - \underline{F}(\tau)\ , \tag{6-6}$$

where

$$\underline{F}(\tau) = \frac{1}{R}\,\langle|\tilde{\Delta}_{\perp}\tilde{A}|^2\rangle + M\langle|\tilde{\Delta}_{\perp}\tilde{\phi}|^2\rangle + \chi\langle|\tilde{\nabla}_{\perp}\tilde{p}|^2\rangle\ .$$

Therefore

$$\frac{d}{d\tau}\,\varepsilon(\tau) = -\underline{F}(\tau)\left(1 + (1+D)\,\frac{\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle}{\underline{F}(\tau)}\right)\ .$$

Let us define D_c as

$$\sup_{\tilde{A},\tilde{\phi},\tilde{p}\,\in\, H}\left(-\frac{\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle}{\underline{F}(\tau)}\right) \equiv \frac{1}{1 + D_c}\ . \tag{6-7}$$

Since $|\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle/\underline{F}|$ is bounded from above, we have $D_c > -1$. In fact, this boundedness is shown as follows. We have

$$\left|\frac{\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle}{\underline{F}(\tau)}\right|^2 \leq \frac{\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\rangle^2}{(M\langle|\tilde{\Delta}_{\perp}\tilde{\phi}|^2\rangle + \chi\langle|\tilde{\nabla}_{\perp}\tilde{p}|^2\rangle)^2}\ . \tag{6-8}$$

From Schwartz's inequality, i.e.

$$\langle \tilde{p} \frac{\partial \tilde{\phi}}{\partial y} \rangle^2 \leq \langle |\tilde{p}|^2 \rangle \left\langle \left| \frac{\partial \tilde{\phi}}{\partial y} \right|^2 \right\rangle \leq \langle |\tilde{p}|^2 \rangle \langle |\tilde{\nabla}_\perp \tilde{\phi}|^2 \rangle$$

and Poincare's inequalities (6-5) for $\tilde{p}$ and $\tilde{\nabla}_\perp \tilde{\phi}$, the right-hand-side of inequality (6-8) is less than

$$C^2 \frac{\langle |\tilde{\nabla}_\perp \tilde{p}|^2 \rangle \langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle}{(M \langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle + \chi \langle |\tilde{\nabla}_\perp \tilde{p}|^2 \rangle)^2} \leq \frac{C^2}{2\chi M}$$

It follows that the supreme value given by Eq. (6-7) can be less than 1 by taking either χ or M to be large enough. Therefore, with such a choice of χ and M, we obtain $D_c > 0$.

Continuing our proof, we get from Eq. (6-7)

$$\frac{d}{d\tau} \varepsilon(\tau) \leq - \underline{F}(\tau) \left(1 - \frac{1+D}{1+D_c}\right) . \tag{6-9}$$

Let $\Lambda = \min(\frac{1}{R}, M, \chi)$, then, by using Poincare's inequalities (6-5) for $\tilde{\nabla}_\perp \tilde{A}$, $\tilde{\nabla}_\perp \tilde{\phi}$ and $\tilde{p}$, we have

$$\begin{aligned} \underline{F}(\tau) &= \Lambda\left(\frac{1}{R\Lambda} \langle |\tilde{\Delta}_\perp \tilde{A}|^2 \rangle + \frac{M}{\Lambda} \langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle + \frac{\chi}{\Lambda} \langle |\tilde{\nabla}_\perp \tilde{p}|^2 \rangle\right) \\ &\geq \Lambda(\langle |\tilde{\Delta}_\perp \tilde{A}|^2 \rangle + \langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle + \langle |\tilde{\nabla}_\perp \tilde{p}|^2 \rangle) \\ &\geq \Lambda C^{-1} \cdot \varepsilon(\tau) . \end{aligned} \tag{6-10}$$

From inequalities (6-9) and (6-10), we obtain

$$\frac{d}{d\tau}\,\varepsilon(\tau) \le -\frac{\Lambda}{C}\,\varepsilon(\tau)\left(1 - \frac{1+D}{1+D_c}\right)$$

or, by integrating, we get

$$\varepsilon(\tau) \le \varepsilon(0)\,\exp\left(-\frac{\Lambda}{C}\left(1 - \frac{1+D}{1+D_c}\right)\tau\right) .$$

Hence, if $D < D_c$, then the perturbation decays exponentially with a decay constant proportional to the smallest dissipation constant $\Lambda = \min(1/R, M, \chi)$. (Q.E.D.)

Since $|\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle/\underline{F}|$ is bounded from above, there exists the least upper bound of $|\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle/\underline{F}|$. Suppose $\tilde{A}_o$, $\tilde{\phi}_o$ and $\tilde{p}_o$ take the maximum value of $-\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle/\underline{F}$. (We note that $\tilde{A}_o$, $\tilde{\phi}_o$ and $\tilde{p}_o$ may not have enough smoothness but they can be approximated by functions in H as "closely" as possible. See Appendix C.) By choosing these $\tilde{A}_o$, $\tilde{\phi}_o$ and $\tilde{p}_o$ as an initial condition, we have, from Eq. (6-7),

$$\frac{d}{d\tau}\,\varepsilon(0) = -\underline{F}(0)\left(1 - \frac{1+D}{1+D_c}\right) .$$

It follows that $d\varepsilon(0)/d\tau \ge 0$ if $D \ge D_c$. Therefore the following corollary holds:

<u>COROLLARY</u>: Under the condition of Theorem 1, define D_c by Eq. (6-7). Then, $D < D_c$ is the necessary and sufficient condition for monotonic stability.

We call this critical value D_c the energy stability limit. We also define the linear stability limit D_L as the value that makes the null solution of the system linearly marginally stable. The following theorem gives the relationship between the energy stability limit D_c and the linear stability limit D_L.

<u>THEOREM 2</u>: Suppose R, M and χ are all positive constants. Then $D_c \leq D_L$.

<u>PROOF</u>: Adding up Eqs. (6-2) and (6-3) and taking the real part of the resulting equation, we obtain

$$0 = (1+D_L)\ \mathrm{Re}\ \langle \tilde{p}\ \frac{\partial \tilde{\phi}^*}{\partial y} \rangle + \frac{1}{R} \langle |\tilde{\Delta}_\perp \tilde{A}|^2 \rangle + M \langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle + \chi \langle |\tilde{p}|^2 \rangle ,$$

where we used the condition that Re q = 0 if $D = D_L$. Therefore we have

$$\frac{1}{1+D_L} = \frac{- \langle \tilde{p}_r \dfrac{\partial \tilde{\phi}_r}{\partial y} + \tilde{p}_i \dfrac{\partial \tilde{\phi}_i}{\partial y} \rangle}{\frac{1}{R} \langle |\tilde{\Delta}_\perp \tilde{A}|^2 \rangle + M \langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle + \chi \langle |\tilde{\nabla}_\perp \tilde{p}|^2 \rangle} . \qquad (6\text{-}11)$$

Here we write $\tilde{p} = \tilde{p}_r + i\tilde{p}_i$ and $\tilde{\phi} = \tilde{\phi}_r + i\tilde{\phi}_i$. Writing $\langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle = \langle |\tilde{\Delta}_\perp \tilde{\phi}_r| \rangle + \langle |\tilde{\Delta}_\perp \tilde{\phi}_i|^2 \rangle$ and $\langle |\tilde{\nabla}_\perp \tilde{p}|^2 \rangle = \langle |\tilde{\nabla}_\perp \tilde{p}_r|^2 \rangle + \langle |\tilde{\nabla}_\perp \tilde{p}_i|^2 \rangle$ and applying the following inequality to Eq. (6-11)

$$\frac{\sum a_n}{C + \sum b_n} \leq \frac{\sum \left(\frac{a_n}{b_n}\right) b_n}{\sum b_n}$$

$$\leq \max \left(\frac{a_n}{b_n}\right) ,$$

where C and b_n are positive, we obtain

$$\frac{1}{1+D_L} \leq \sup_{\tilde{\phi},\tilde{p} \in H} \frac{- \langle \tilde{p} \frac{\partial \tilde{\phi}}{\partial y} \rangle}{M \langle |\tilde{\Delta}_\perp \tilde{\phi}|^2 \rangle + \chi \langle |\tilde{\nabla}_\perp \tilde{p}|^2 \rangle}$$

$$= \sup_{\tilde{A},\tilde{\phi},\tilde{p} \in H} - \frac{\langle \tilde{p} \frac{\partial \tilde{\phi}}{\partial y} \rangle}{\underline{F}(\tau)}$$

$$= \frac{1}{1+D_c} .$$

It follows that $D_c \leq D_L$. (Q.E.D.)

To conclude this section, it is found that we need positive parameters of D as well as R, χ, M in the system (T) in order to find the solutions that grow from small initial values and eventually saturate with finite amplitude or lead to stationary turbulence. It is also found that, with a proper choice of the viscosity M and the heat conductivity χ, there exists a positive energy stability limit D_c, which is smaller than or equal to the linear stability limit D_L. This

shows that, if the plasma is sufficiently viscous or heat conductive, the null solution is stable (to any perturbation) even for some positive D. Since D is the only free energy source of the system, if $D < D_c$, then all the energy fed to the modes by the mean pressure gradient dissipates through the collisional diffusion, and not through convection. When $D > D_L$, then even an infinitesimal perturbation given to the null solution starts to grow and the free energy produced from the mean pressure gradient is transfered by convective motion of the plasma. In the next section, we will consider the behavior of such convection near the linear stability limit D_L.

VII. BIFURCATION ANALYSIS OF A TOKAMAK PLASMA

In this section, we will consider the nonlinear behavior of perturbations of a linearly unstable null solution and we will derive the dependence of the anomalous heat transport on the prameter D, which is proportional to the mean pressure gradient. As discussed in the previous section, if the parameter D is larger than the linear stability limit D_L, then modes on the rational surface begin to grow even for an infinitesimal initial perturbation. It is possible, however, that the given perturbation eventually saturates with a finite amplitude. In this case, the sum of the null solution and this saturated mode gives us another equilibrum state, a stable one, which bifurcates from the unstable null solution. It is also possible that a small perturbation given at $\tau = 0$ developes into turbulence if the pressure gradient D is sufficiently larger than the linear stability limit D_L. In the following, we consider the former case: D is slightly larger than D_L so that we expect to have steady convection within the boundary layer, which is analogous to the Benard convection[16] in conventional fluid dynamics.

For simplicity, we will assume that mode rational surfaces in a plasma are well separated from each other and mode-mode interactions between two different magnetic surfaces are ignorable. This is certainly not the case for a turbulent plasma but this is also a reasonable assumption for a certain region of a relatively quiet tokamak plasma. Under this assumption, we will consider nonlinear modes on a single rational surface, where the modes with the same helicity interacts with each other nonlinearly. These modes on a single rational surface can be described as the $\bar{z}$-independent solutions

of the nonlinear system (T), where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are the coordinates defined as $\tilde{x} = x$, $\tilde{y} = y + \hat{\theta}x$ and $\tilde{z} = \tilde{\theta}$ in terms of coordinates $(x, y, \tilde{\theta})$ used in this system (see Sections IV and V.) We will use the coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ throughout this section.

The $\tilde{z}$-independent normal mode solutions of the linearized system of Eqs. (T) have the forms

$$\begin{aligned} \tilde{A}(\tilde{x}, \tilde{y}) &= \tilde{A}(\tilde{x}) \exp(imk\tilde{y}) \\ \tilde{\phi}(\tilde{x}, \tilde{y}) &= \tilde{\phi}(\tilde{x}) \exp(imk\tilde{y}) \\ \tilde{p}(\tilde{x}, \tilde{y}) &= \tilde{p}(\tilde{x}) \exp(imk\tilde{y}) , \end{aligned} \qquad (7\text{-}1)$$

if we impose the boundary condition (II'), instead of (II), given in Section IV. Here m is an integer, and $k = \pi/\delta_y$ (δ_y is defined in Section IV). The helicity of these modes is given by the pitch length μ in Eqs. (3-17) at this magnetic surface. In fact, we have[14] $\tilde{y} \propto (z-\mu\theta)$ to the lowest order, where θ and z are the azimuthal (or poloidal) direction and the longitudinal (or toroidal) direction of the cylindrical plasma, respectively. Since δ_y is a quantity of the order of $1/\sqrt{R}$, where R is the magnetic Reynolds number, we note that the modes given in Eqs. (7-1) with m of order 1 are the modes with short wavelengths, which correspond to the resistive fast interchange modes.[8]

The $\tilde{z}$-independent solutions of the system (T) are, therefore, a nonlinear extension of the linear resistive fast interchange modes. In particular, the $\tilde{z}$-independent steady solutions are the extension of the linear marginal modes of the system. We now consider such steady solutions on a single rational surface. When the parameter D, which is the free energy source of the modes, is slightly larger than the linear

stability limit D_L, a perturbation growing from an infinitesimal initial value is expected to saturate with small amplitude. In this case, we are able to obtain the expressions of the nonlinearly saturated modes and to estimate the anomalous heat transport caused by these modes, by expanding the solutions in terms of the small amplitude.[11]

Let ε^2 be the energy of the perpendicular flow of such a solution, i.e.,

$$\varepsilon^2 = \frac{1}{2} \langle |\tilde{\nabla}_\perp \tilde{\phi}|^2 \rangle . \tag{7-2}$$

Assuming that ε is a small number, we introduce new dependent variables $\hat{A}$, $\hat{\phi}$ and $\hat{p}$ defined by $\tilde{A} = \varepsilon\hat{A}$, $\hat{\phi} = \varepsilon\hat{\phi}$ and $\tilde{p} = \varepsilon\hat{p}$. Since we consider the steady $\bar{z}$-independent solutions, we have $\partial/\partial\tau = 0$ and $\partial/\partial\tilde{z} = 0$ (i.e., $\partial/\partial\tilde{\theta} = \tilde{x}\partial/\partial\tilde{y}$). Therefore, Eqs. (T) become

$$\varepsilon\{\hat{\phi},\hat{A}\} = \bar{x} \frac{\partial}{\partial\tilde{y}} \hat{\phi} + \frac{1}{R} \tilde{\Delta}_\perp \hat{A} \tag{7-3a}$$

$$\varepsilon\{\hat{\phi},\tilde{\Delta}_\perp\tilde{\phi}\} = \tilde{x} \frac{\partial}{\partial\tilde{y}} \tilde{\Delta}_\perp \hat{A} + \varepsilon\{\hat{A},\tilde{\Delta}_\perp \hat{A}\} - \frac{\partial\hat{p}}{\partial\tilde{y}} + M\tilde{\Delta}_\perp^2\hat{\phi} \tag{7-3b}$$

$$\varepsilon\{\hat{\phi},\hat{p}\} = - D \frac{\partial\hat{\phi}}{\partial\tilde{y}} + \chi\tilde{\Delta}_\perp \hat{p} . \tag{7-3c}$$

Our aim in this section is to solve the system (7-3), together with Eq. (7-2), assuming that ε is a small quantity. As boundary conditions, we

take (II'), instead of (II), as well as (I) and (III) given in Section IV, so that the solutions of this system are the natural extensions of the linear marginal modes.

In the system (7-3), all the coefficients depend on ε analytically: in fact, some coefficients do not depend on ε and the others are just ε's. Therefore the solutions $\hat{A}$, $\hat{\phi}$ and $\hat{p}$ of this system depend on ε analytically if they exist at all. However the existence of the nontrivial solutions of this system is not guaranteed for all ε (for fixed other parameters D, R, M and χ). In fact, if $\varepsilon = 0$, then the system (7-3) forms an eigenvalue problem and that $D = D_L$ is the condition that the nontrivial solutions of this linear system exist. Thus, for a finite ε, we choose a proper $D = D(\varepsilon)$ as a function of ε (whereas the other parameters R, M and χ remain as given constants) so that we expect the solutions of the system (7-3) to exist for any (small) ε. Here we assume the analytic dependence of D on ε and expand the solutions and D in power series of ε as follows:

$$\hat{A} = \sum_{n=0}^{\infty} \hat{A}_n \varepsilon^n , \qquad \hat{\phi} = \sum_{n=0}^{\infty} \hat{\phi}_n \varepsilon^n , \qquad \hat{p} = \sum_{n=0}^{\infty} \hat{p}_n \varepsilon^n$$

and

$$D = D_L + \sum_{n=1}^{\infty} D_n \varepsilon^n .$$

We now seek formal solutions of this type by substituting these quantities into Eqs. (7-3). Considering the case where $D_L > 0$ (this choice is always possible as shown in Section VII), we obtain the following system as the zeroth order system in ε:

$$L \begin{bmatrix} \hat{\phi}_o \\ \\ \hat{p}_o \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \tag{7-4}$$

where L is a linear operator defined as

$$L \begin{bmatrix} \hat{\phi}_o \\ \\ \hat{p}_o \end{bmatrix} = \begin{bmatrix} -R\tilde{x}^2 \dfrac{\partial^2 \hat{\phi}_o}{\partial \tilde{y}} - \dfrac{\partial \hat{p}_o}{\partial \tilde{y}} + M\tilde{\Delta}_\perp^2 \hat{\phi}_o \\ \dfrac{\partial \hat{\phi}_o}{\partial \tilde{y}} - \dfrac{\chi}{D_L} \tilde{\Delta}_\perp \hat{p}_o \end{bmatrix} \tag{7-5}$$

and $\hat{A}_o$ is determined from

$$\tilde{\Delta}_\perp \hat{A}_o = -R\tilde{x} \frac{\partial \hat{\phi}_o}{\partial \tilde{y}} . \tag{7-6}$$

The lowest order term of Eq. (7-2) determines the amplitude of $\hat{\phi}_o$:

$$\langle |\tilde{\nabla}_\perp \hat{\phi}_o|^2 \rangle = 2 . \tag{7-7}$$

We now consider marginally stable solutions having the following forms:

$$\begin{aligned} \hat{A}_o &= \hat{A}_{01}(\tilde{x}) \cos k\tilde{y} , \\ \hat{\phi}_o &= \hat{\phi}_{01}(\tilde{x}) \sin k\tilde{y} , \\ \hat{p}_o &= \hat{p}_{01}(\tilde{x}) \cos k\tilde{y} , \end{aligned} \tag{7-8}$$

where k is a given wave number in the $\tilde{y}$-direction. In this case, Eq. (7-4) becomes

$$L_k \begin{bmatrix} \hat{\phi}_{01} \\ \\ \hat{p}_{01} \end{bmatrix} = \begin{bmatrix} Rk^2\tilde{x}^2 + M(\partial_x^2-k^2)^2 & k \\ \\ k & -\frac{X}{D_L}(\partial_x^2-k^2) \end{bmatrix} \begin{bmatrix} \hat{\phi}_{01} \\ \\ \hat{p}_{01} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \tag{7-9}$$

Here $\partial_x = \partial/\partial\tilde{x}$, and the functions $\hat{\phi}_{01}$ and $\hat{p}_{01}$ are taken to be real-valued functions of $\tilde{x}$. We think of L_k as an operator applied to a pair of real-valued functions $(u_1,u_2)^t$ of $\tilde{x}$, where t denotes the transpose of the vector and u_1 and u_2 are sufficiently smooth L^2-functions. We define the inner product of such functions $(u_1,u_2)^t$ and $(v_1,v_2)^t$ as

$$(u_1,u_2) \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} = \langle u_1v_1 + u_2v_2 \rangle$$

It is clear that L_k is a Hermitian operator with respect to this inner product.

In Eq. (7-9), we note that the linear stability limit D_L is a function of k. In order not to generalize the problem too much, we here choose the wave number k such that the mode in Eqs. (7-8) is the only marginal mode, in other words, all the other modes whose $\tilde{y}$-dependence is given by either $\sin \ell k\tilde{y}$ or $\cos \ell k\tilde{y}$ ($\ell \neq 1$) have negative linear growth rate when $D = D_L$. Once we choose such a wave

number k, then we fix this k and D_L, and treat them as constants throughout our discussion. In this case, the operator $L_{\ell k}$, where ℓ is an integer, is always invertible unless $\ell = 1$, so that the linear equation $L_{\ell k}(u_1,u_2)^t = (0,0)^t$ only gives the trivial solution if $\ell \neq 1$.

Thus, by solving the linear homogeneous equation (7-6) and Eq. (7-9) with the normalization condition (7-7), we obtain the solutions $\hat{A}_o$, $\hat{\phi}_o$, and $\hat{p}_o$ (in the form of Eqs. (7-8)) of the zeroth order system (7-4). With these solutions, the first order system in ε from Eqs. (7-1) and (7-3) becomes

$$\langle \tilde{\nabla}_\perp \hat{\phi}_o \cdot \tilde{\nabla}_\perp \hat{\phi}_1 \rangle = 0 \ , \tag{7-10}$$

$$L \begin{bmatrix} \hat{\phi}_1 \\ \\ \hat{p}_1 \end{bmatrix} = \begin{bmatrix} \{\hat{\phi}_o, \tilde{\Delta}_\perp \hat{\phi}_o\} - R\tilde{x} \dfrac{\partial}{\partial \tilde{y}} \{\hat{\phi}_o, \hat{A}_o\} - \{\hat{A}_o, \tilde{\Delta}_\perp \hat{A}_o\} \\ \\ \left(\{\phi_o, p_o\} + D_1 \dfrac{\partial \hat{\phi}_o}{\partial \tilde{y}}\right)/D_L \end{bmatrix} \tag{7-11}$$

$$\tilde{\Delta}_\perp \hat{A}_1 = -R\tilde{x} \frac{\partial}{\partial \tilde{y}} \hat{\phi}_1 + R\{\hat{\phi}_o, \hat{A}_o\} \ . \tag{7-12}$$

If we expand the solutions $\hat{A}_1$, $\hat{\phi}_1$ and $\hat{p}_1$ in Fourier series in $\tilde{y}$ such as $\hat{A}_1 = \sum_\ell \hat{A}_{1\ell} \cos \ell k y$, $\hat{\phi}_1 = \sum_\ell \hat{\phi}_{1\ell} \sin \ell k y$ and $\hat{p}_1 = \sum_\ell \hat{p}_{1\ell} \cos \ell k y$, it is clear that we obtain only three sets of equations for $\ell = 0,1$ and 2 from Eqs. (7-11) and (7-12).

For $\ell = 1$, we have

$$L_k \begin{bmatrix} \hat{\phi}_{11} \\ \\ \hat{p}_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ k \dfrac{D_1}{D_L} \hat{\phi}_{01}(\tilde{x}) \end{bmatrix} \tag{7-13}$$

The necessary and sufficient condition that Eq. (7-13) have a nontrivial solution is that the right-hand-side is orthogonal to the null space of the adjoint operator L_k^+ of L_k. This condition determines D_1. Since $L_k = L_k^+$, we only have to require that the inner product of $(\hat{\phi}_{01}, \hat{p}_{01})$ and the right-hand-side of Eq. (7-13) be equal to zero:

$$k \frac{D_1}{D_L} \langle \hat{p}_{01} \hat{\phi}_{01} \rangle = 0 \; . \tag{7-14}$$

Multiplying the second equation of Eq. (7-9) by $\hat{p}_{01}$ and taking the $\langle\ \rangle$ average of the resulting equation yields

$$k \langle \hat{p}_{01} \hat{\phi}_{01} \rangle = - \frac{\chi}{D_L} \langle (\partial_x \hat{p}_{01})^2 + k^2 \hat{p}_{01}^2 \rangle \; ,$$

so $\langle \hat{p}_{01} \hat{\phi}_{01} \rangle$ is nonzero for nonzero $\hat{p}_{01}$. Therefore, from Eq. (7-14), $D_1 = 0$ and the solution of Eq. (7-13) becomes

$$\begin{bmatrix} \hat{\phi}_{11} \\ \\ \hat{p}_{11} \end{bmatrix} = C \begin{bmatrix} \hat{\phi}_{01} \\ \\ \hat{p}_{01} \end{bmatrix} \tag{7-15}$$

with some constant C. For $\ell = 0$ and 2, we obtain $(\hat{\phi}_{10}, \hat{p}_{10})^t$ and $(\hat{\phi}_{12}, \hat{p}_{12})^t$ from Eq. (7-11) by just inverting the opertors L_o and L_{2k}

since the null spaces of these operators consist of only one element $(0,0)^t$. By taking into account Eq. (7-10), i.e.,

$$\langle \tilde{\nabla}_\perp \hat{\phi}_{01} \cdot \tilde{\nabla}_\perp \hat{\phi}_{11} \rangle = 0 \ ,$$

we have $C = 0$ in Eq. (7-15) and the solution of Eq. (7-10) is given by

$$\begin{bmatrix} \hat{\phi}_1 \\ \\ \hat{p}_1 \end{bmatrix} = \begin{bmatrix} \hat{\phi}_{10} \\ \\ \hat{p}_{10} \end{bmatrix} + \begin{bmatrix} \hat{\phi}_{12} \sin 2k\tilde{y} \\ \\ \hat{p}_{12} \cos 2k\tilde{y} \end{bmatrix} \tag{7-16}$$

The second order system in ε from Eqs. (7-3) is

$$L \begin{bmatrix} \hat{\phi}_2 \\ \\ \hat{p}_2 \end{bmatrix} = \begin{bmatrix} f \\ \\ -(g - D_2 \dfrac{\partial \phi_2}{\partial y})/D_L \end{bmatrix} \tag{7-17}$$

and

$$\tilde{\Delta}_\perp \hat{A}_2 = -R\bar{x} \frac{\partial}{\partial \tilde{y}} \hat{\phi}_2 + h$$

where

$$f = \{\hat{\phi}_o,\tilde{\Delta}_\perp\hat{\phi}_1\} + \{\hat{\phi}_1,\tilde{\Delta}_\perp\hat{\phi}_o\} - \{\hat{A}_o,\tilde{\Delta}_\perp\hat{A}_1\} - \{\hat{A}_1,\tilde{\Delta}_\perp\hat{A}_o\} - x\frac{\partial}{\partial y}h$$

$$g = \{\hat{\phi}_o,\hat{p}_1\} + \{\hat{\phi}_1,\hat{p}_o\}$$

and

$$h = R(\{\hat{\phi}_o,\hat{A}_1\} + \{\hat{\phi}_1,\hat{A}_o\}) \ .$$

We solve Eq. (7-17) by using the following Fourier expansion in $\tilde{y}$: $\hat{A}_2 = \sum_\ell \hat{A}_{2\ell} \cos \ell k\tilde{y}$, $\hat{\phi}_2 = \sum_\ell \hat{\phi}_{2\ell} \sin \ell k\tilde{y}$, $\hat{p}_2 = \sum_\ell \hat{p}_{2\ell} \cos \ell k\tilde{y}$, $f = \sum_\ell f_\ell \sin \ell k\tilde{y}$ and $g = \sum_\ell g_\ell \cos \ell k\tilde{y}$. For $\ell = 1$, we have

$$L_k \begin{bmatrix} \hat{\phi}_{21} \\ \\ \hat{p}_{21} \end{bmatrix} = \begin{bmatrix} f_1 \\ \\ -(g_1+kD_2\hat{\phi}_{01})/D_L \end{bmatrix} , \tag{7-18}$$

the solvability condition of which determines D_2, similar to the case of Eq. (7-13). The necessary and sufficient condition that Eq. (7-18) has a solution is, as before, that the inner product of $(\hat{\phi}_{01},\hat{p}_{01})$ and the right-hand-side of Eq. (7-18) be equal to zero. Therefore, D_2 is given by

$$D_2 = (D_L\langle\hat{\phi}_{01}f_1\rangle - \langle\hat{p}_{01}g_1\rangle)/k\langle\hat{p}_{01}\hat{\phi}_{01}\rangle \ . \tag{7-19}$$

Assuming that this D_2 is not zero, we are able to relate the anomalous heat transport $\langle s_1 v_{1r}\rangle$ to the parameter D. Since $\langle\tilde{p}\partial\tilde{\phi}/\partial\tilde{y}\rangle = \varepsilon^2 k\langle\hat{p}_{01}\hat{\phi}_{01}\rangle$ and $D = D_L + \varepsilon^2 D_2$ to the lowest order of ε, we have

$$\langle \tilde{p} \frac{\partial \tilde{\phi}}{\partial \tilde{y}} \rangle = k\langle \hat{p}_{01}\hat{\phi}_{01}\rangle \frac{(D-D_L)}{D_z}$$

or

$$- p_o \langle s_1 v_{1r}\rangle = \frac{B_\theta B^2 \sigma^2}{(\gamma-1)\sqrt{\rho_o}} k\langle \hat{p}_{01}\hat{\phi}_{01}\rangle \frac{(D-D_L)}{D_2} \qquad (7\text{-}20)$$

near $D = D_L$. Here we used Eqs. (3-20) and (3-26) to relate $\langle \tilde{p}\partial\tilde{\phi}/\partial\tilde{y}\rangle$ to $\langle s_1 v_{1r}\rangle$. Equation (7-20) gives the dependence of the anomalous heat flux $-p_o\langle s_1 v_{1r}\rangle$ on the parameter D, which we have been after in this section.

In summary, we have derived the dependence of the anomalous heat flux on the parameter D, which is proportional to the mean pressure gradient p_o'. Since this derivation is based on a small amplitude expansion, the result is correct only when D is slightly larger than the linear stability limit D_L. In this region, it is found that the anomalous heat flux is linear with $(D-D_L)$. The dependence on other parameters R, M and χ are, however, not apparent in our results since the expression of the anomalous heat transport given in Eq. (7-20) is a functional of the linear marginal modes, which depend on R, M and χ implicitly.

When the parameter D is much larger than the linear stability limit D_L, the nonlinearity of the system is so strong that we need numerical calculations to estimate the anomalous heat transport. In the next section, we will show some results of the numerical simulations and discuss the validity of the analytical results of Eq. (7-20).

VIII. NUMERICAL CALCULATIONS OF THE REDUCED EQUATIONS

In this section, the results of numerical calculations of the reduced equations are presented and some properties of the anomalous heat transport are discussed. Here we only deal with the system (T), i.e., a tokamak plasma, in which the anomalous heat transport is decoupled from the dynamo effect. It is shown that steady convection within a boundary layer is attained and the anomalous heat transport arising from this convection weakly depends on the collisional heat conductivity. Therefore, when the effect of the collisional heat conduction is small, the total heat transport is dominated by the anomalous one caused by the convective cells. It is also shown that the analytical estimation of the anomalous heat transport given by Eq. (7-20) agrees well with the results of the numerical calculations near the linear stability limit D_L.

The nonlinear initial value code to solve the reduced equations (T) is developed from the HIB code of Strauss.[19,20] The boundary conditions employed in our numerical code are (I), (II') and (III) given in Section IV; the equations are solved in the domain of the slab geometry, i.e. $|\tilde{x}| \leq \delta_x$, $|\tilde{y}| \leq \delta_y$ and $|\tilde{z}| \leq L_z$, which corresponds to a boundary layer in the plasma. The numerical method used in the code is a finite difference scheme for the variable $\tilde{x}$, Fourier component representations for the variables $\tilde{y}$ and $\tilde{z}$, and the predictor-corrector method[21] for time τ. Since we consider the fluctuations on a single rational surface, the $\tilde{z}$-dependence of the solutions is ignored, and nonlinear interactions of the modes with the same helicity are calculated. Moreover, for simplicity, only the Fourier cosine components for $\tilde{\phi}$ and $\tilde{p}$ and the Fourier sine components of $\tilde{A}$ are taken

into consideration: $\tilde{\phi} = \sum_m \hat{\phi}_m(\tilde{x}) \cos(m\tilde{y}/y_o)$, $\tilde{p} = \sum_m \hat{p}_m(\tilde{x}) \cos(m\tilde{y}/y_o)$, and $\tilde{A} = \sum_m \hat{A}_m(\tilde{x}) \sin(m\tilde{y}/y_o)$, where $y_o = \delta_y/\pi$. Therefore, as in the previous section, the modes considered here are a nonlinear extension of the normal mode representations of the linear resistive g-mode.

Schematically, the numerical algorithm of the system (T) employed in our code is summarized as follows:

$$\hat{A}^*_m = \hat{A}^n_m + h\Delta\tau(-[\tilde{\phi}^n, A^n]_m + \tilde{x}\frac{m}{y_o}\hat{\phi}^n_n + \frac{1}{R}\Delta_\perp \hat{A}^*_m)$$

$$\hat{W}^*_m = \hat{W}^n_m + h\Delta\tau(-[\tilde{\phi}^n, \tilde{W}^n]_m + [\tilde{A}^n, \tilde{\Delta}_\perp \tilde{A}^n]_m - \tilde{x}\frac{m}{y_o}\Delta_\perp \hat{A}^n_m + \frac{m}{y_o}\hat{p}^n_m + M\Delta_\perp \hat{W}^*)$$

$$\hat{p}^*_m = \hat{p}^n_m + h\Delta\tau(-[\tilde{\phi}^n, \tilde{p}^n]_m - D\frac{m}{y_o}\hat{\phi}^n_m + \chi\Delta_\perp \hat{p}^*_m)$$

$$\hat{\phi}^*_m = \Delta_\perp \hat{W}^*_m$$

$$\hat{A}^{n+1}_m = \hat{A}^n_m + \Delta\tau(-[\tilde{\phi}^*, A^*]_m + \tilde{x}\frac{m}{y_o}\hat{\phi}^*_m + \frac{1}{R}\Delta_\perp \hat{A}^{n+1}_m)$$

$$\hat{W}^{n+1}_m = \hat{W}^n_m + \Delta\tau(-[\tilde{\phi}^*, \tilde{W}^*]_m + [\tilde{A}^*, \tilde{\Delta}_\perp \tilde{A}^*]_m - \tilde{x}\frac{m}{y_o}\Delta_\perp \hat{A}^*_m + \frac{m}{y_o}\hat{p}^*_m + M\Delta_\perp \hat{W}^{n+1}_m)$$

$$\hat{p}^{n+1}_m = \hat{p}^n_m + \Delta\tau(-[\tilde{\phi}^*, \tilde{p}^*]_m - D\frac{m}{y_o}\hat{\phi}^*_m + \chi\Delta_\perp \hat{p}^{n+1}_m)$$

$$\hat{\phi}^{n+1}_m = \Delta_\perp \hat{W}^{n+1}_m \; .$$

Here $\Delta\tau$ denotes the time step size, $\tilde{A}^n$ and $\hat{A}^n_m$ denote $\tilde{A}$ and $\hat{A}_m(\tilde{x})$ at n-th time step, respectively, so that $\tilde{A}^n = \sum_m \hat{A}^n_m \sin(m\tilde{y}/y_o)$, $[\ ,\ ]_m$ indicates the m-th Fourier component of $[\ ,\ ]$ and $\Delta_\perp = d^2/d\tilde{x}^2 - (m/y_o)^2$. The intermediate time step for the predictor-corrector advance is chosen by taking h to be 0.65. Since the implicit time-difference scheme is employed for the operator $\Delta_\perp$ in the right-hand-sides of these equations, operators of the form $\Delta_\perp + c$, where c is some constant, are inverted at every time step to solve these equations.

This implicit scheme allows us to avoid the Courant-Friedrichs-Lewy condition[22] and to take a larger step size $\Delta\tau$.

We now take the small scale parameter δ introduced in Section III to be $1/\sqrt{R}$, where R is the magnetic Reynolds number. In terms of this δ, the following size of the domain is used for our boundary conditions: $\delta_x/\delta = 50$ and $\delta_y/\delta = 2\pi$ (therefore $y_o = 2\delta$ in the equations above). As we will see later, the solutions of these equations decay rapidly as $|\tilde{x}| \to \delta_x$. It is also confirmed numerically that these solutions hardly depend on the choice of δ_x if δ_x/δ is taken to be 50 or larger. Therefore we use the averaging operator $\langle\ \rangle$ defined in Eq. (4-3) with $\Delta = \delta_y$ in order to estimate various averaged quantities numerically. In fact, it is observed in these numerical computations that the typical mode width in the $\tilde{x}$-direction Δ is of the same order of the one in the $\tilde{y}$-direction δ_y.

The boundary conditions (I) and (II') imposed here depend on the magnetic Reynolds number R since δ_x and δ_y are proportional to $\delta = 1/\sqrt{R}$. However after the scale transformation (3-23) is applied to this system, this dependence disappears. As we discussed after Eq. (3-24) in Section III, therefore, the solutions of this transformed system i.e. $\tilde{A}/\delta^2$, $\tilde{\phi}/\delta^2$ and $\tilde{p}/\delta$, depend only on the parameters D, M_R ($= \mu_\perp/\eta\rho_o$) and K_R ($= (\gamma-1)\kappa_\perp/\eta\rho_o$) but not on R. Here we call M_R and K_R the normalized viscosity and the normalized heat conductivity, respectively. Therefore, in actual numerical calculations, we solve this transformed system and leave the magnetic Reynolds number R as an undetermined parameter.

Typically we use the following values of the parameters in our calculations unless otherwise specified: $D = 0.20$ and $M_R, K_R = 1 - 10^{-2}$.

Here we note that $D < 0.25$ is the ideal stability condition by Suydam.[13] We also note that $M_R \sim K_R \sim \beta \sqrt{m_i/m_e}$, where β is the plasma beta, m_i and m_e are the masses of the ion and the electron, respectively, if we take $\mu_\perp$ and $\kappa_\perp$ in Eqs. (2-1) to be the ion viscosity and the ion heat conductivity, respectively, in a collisional plasma given in Ref. 12. As before, the parallel heat conductivity $\chi_{||}$ is taken to be smaller than $O(1)$ so that it does not enter the equations. However we point out that some preliminary calculations including $\chi_{||}$ show that $\chi_{||}$ has a stabilizing effect and thus reduces somewhat the anomalous heat transport across the magnetic surface. In most runs presented here, 150 mesh ponts in the $\tilde{x}$-direction and 7 modes $(0 \leq m \leq 6)$ in the $\tilde{y}$-direction are employed. It is confirmed by varying the numbers of the mesh points and the modes that this is a sufficient numerical resolution to obtain the correct saturation levels of the convection. The initial condition given to the calculations is a sufficiently small perturbation of the $m = 1$ component of $\tilde{\phi}$.

Figures 1(a)-1(e) show the Fourier components of $\tilde{A}$, $\tilde{\Delta}_\perp \tilde{A}$, $\tilde{\phi}$, $\tilde{\Delta}_\perp \tilde{\phi}$ and $\tilde{p}$, respectively, of the typical steady convection of a tokamak obtained from Eqs. (T). The dominant mode is the $m = 1$ mode for all the quantities except for $\tilde{p}$, whose dominant mode is the $m = 0$ mode. It is seen, however, that there are many large modes other than the $m = 1$ mode, particularly in the current density $\tilde{\Delta}_\perp \tilde{A}$, the vorticity $\tilde{\Delta}_\perp \tilde{\phi}$ and the pressure $\tilde{p}$. Therefore the nonlinearity of the system plays an important role in the state of the steady convection. The dominance of the $m = 0$ mode of $\tilde{p}$ shows the flattening of the mean pressure gradient, which is what one expects a pressure-driven instability to do. Figure 2 shows the level curves of $\tilde{\phi}$, i.e. the stream lines of the

convection, corresponding to Fig. 1(c). It is seen from these figures that, as we have expected, the modes decay rapidly as $|\bar{x}| \to \infty$ and the typical widths of the mode in the $\bar{x}$-direction and in the $\bar{y}$-direction are of the same order.

The anomalous heat transport across the magnetic surface is often described by the Nusselt number,

$$Nu = 1 + \frac{-p_0\langle s_1 v_{1r}\rangle}{\kappa_\perp T_0'} . \qquad (8\text{-}1)$$

This is the ratio of the actual heat flux, i.e. the sum of the collisional heat flux $\kappa_\perp T_0'$ and the anomalous heat flux $-p_0\langle s_1 v_{1r}\rangle$ due to the convection, to the collisional heat flux $\kappa_\perp T_0'$. We note that the Nusselt number Nu is invariant under the scale transformation (2-23), so that Nu also depends only on D, M_R and K_R. Figure 3 shows the time evolution of the Nusselt number of the modes whose final state is described in Figs. 1(a)-1(e). It is seen that the modes grow linearly at the initial stage and reach the steady convection state after about $100\tau_A$, where τ_A is the Alfven time defined as $r_0\sqrt{\rho_0}/(B_\theta|\sigma|)$ [see Eqs. (3-16)]. The curves (b) in Figs. 4 and 5 show the dependence of the Nusselt number on the parameter D obtained from the simulation. In Fig. 4, the normalized heat conductivity $K_R = 10^{-2}$, and in Fig. 5, $K_R = 10^{-1}$. The two straight lines (a) in these figures describe our analytical results, calculated from Eq. (7-20). In these calculations, the constant D_2 in Eq. (7-20) was obtained through Eq. (7-19) by solving Eq. (7-11) numerically. The linear marginal mode solutions appearing in the right-hand side of Eq. (7-11) were obtained

from the linear version of the initial value code for Eqs. (T). It is seen that these Nusselt numbers obtained from the bifurcation analysis are in good agreement with the results of the nonlinear simulation near the critical point D_L.

We now seek the dependence of the Nusselt number Nu on the normalized diffusion coefficients M_R and K_R for a fixed D. Figure 6 shows that the anomalous heat flux $-p_o\langle s_1 v_{1r}\rangle$ varies weakly with normalized heat conductivity K_R. Here the nondimensional quantity J_a denotes the anomalous heat flux $-p_o\langle s_1 v_{1r}\rangle$ measured by the quantity $n\rho_o T_o'$, i.e., $J_a = -p_o\langle s_1 v_{1r}\rangle/(n\rho_o T_o')$. This weak dependence of J_a on K_R is the reason that the Nusselt number Nu increases with K_R^{-1} almost linearly. In a realistic parameter range for fusion experiments such that $D = 0.2$ and M_R, $K_R = 1.0 - 10^{-2}$, it is seen that the Nusselt number Nu varies from 1 to 10, in other words, the steady convection enhances the heat transport up to one order of magnitude over the collisional heat conduction. It is also seen from Fig. 6 that when K_R is less than 10^{-2}, the anomalous heat flux or the saturation level of the modes increases as some power of K_R^{-1}. This corresponds to the fact that there is no saturated mode nor stationary turbulence when $K_R = 0$, which was stated and proved as Proposition 2 in Section VI. Figure 7 shows that the anomalous heat flux or the Nusselt number Nu with fixed K_R also depends weakly on the normalized viscosity M_R.

IX. DISCUSSIONS AND CONCLUSIONS

In this thesis, we have considered the anomalous heat transport caused by the resistive g-mode fluctuations in a plasma. The resistive g-mode is the instability caused by pressure or temperature gradients acting against the curvature of the magnetic field lines in a plasma with finite resistivity, analogous to the thermal instability causing the Benard convection in a fluid heated form below in a gravity field. We have derived two systems of nonlinear equations describing such fluctuations, one for an RFP plasma and the other for a tokamak plasma, under the assumption that the fluctuating quantities have much smaller scales in space and time than the corresponding mean quantities. In these equations, we take into account the effects of all the collisional diffusion coefficients, i.e. resistivity, viscosity and heat conductivity. These systems are an extension of the reduced equations in Ref. 1.

The inclusion of all the collisional diffusion coefficients in the reduced equations allows us to derive some relations between different kinds of anomalous transport. It is shown based on these reduced equations that the anomalous heat transport $\langle s_1 \underset{\sim}{v}_1 \rangle$ is related to the dynamo effect $\underset{\sim}{\varepsilon}$ in the RFP plasma through Eq. (4-10), i.e., $\underset{\sim}{\varepsilon} \cdot \underset{\sim}{J}_o + \rho_o \langle \underset{\sim}{s}_1 \underset{\sim}{v}_1 \rangle \cdot \nabla T_o = -A^2$, where A is the rate of the energy dissipation of the fluctuations. Therefore one should expect that there is a large anomalous heat transport when there is a strong dynamo activity in the RFP. On the other hand, it is also shown that, in the tokamak, the dynamo effect is smaller than the anomalous heat transport by an order of the inverse aspect ratio. In this case, the anomalous heat flux can be written as $-K^2 \nabla T_o$ by using a non-negative function K^2,

which may depend on the mean quantities, including ∇T_0. In other words, the anomalous transport behaves like the collisional heat conduction in the sense that they both transport heat in the direction opposite to the temperature gradient.

For the resistive g-mode, the free-energy source is the mean pressure (or temperature) gradient and the energy sink is the collisional diffusion. Therefore when the mean pressure gradient exceeds a critical value, modes localized on a rational surface begin to grow. However these modes eventually saturate with finite amplitude if the mean pressure gradient is not too large compared to the critical value, and we have steady convection within a boundary layer which is analogous to the Benard convection in fluid dynamics. This similarity between the resistive g-mode and the Benard convection is examined based on the system (T) of equations for a tokamak plasma.

Applying the nonlinear stability analysis to the system (T), we have shown that there exists a positive critical value D_c of the parameter D, where D is proportional to the mean pressure gradient, such that all perturbations with arbitrary amplitude decay monotonically in time if $D < D_c$. In fact, this D_c is obtained as the energy stability limit, or the necessary and sufficient condition for monotonic stability. It is also shown that the linear stability limit D_L of the system (T) is larger than or equal to the energy stability limit D_c. Since all perturbations with arbitrarily small amplitude grow at least initially if $D > D_L$, and these perturbations are expected to saturate eventually with finite amplitude, we see the similarity between the resistive g-mode of the system (T) and the Benard

convection in fluid dynamics. Here we note that the parameter D corresponds to the Rayleigh number[16] of the Benard convection.

By regarding the steady convection as a bifurcation from the null solutions of the reduced equations, we apply the nonlinear bifurcation analysis to the system (T) in order to derive the dependence of the anomalous heat transport on the mean pressure gradient or the parameter D. In this method, the reduced equations are expanded in terms of the small amplitude, and the complete algorithm to determine all the higher order terms is obtained. It is shown by this method that, to the lowest order, the anomalous heat transport is proportional to the difference of the parameter D and its linear stability limit D_L, that is, $(D-D_L)$.

The validity of the analysis mentioned above is confirmed by using the numerical calculations of the nonlinear reduced equations. These numerical calculations also determine the dependence of the anomalous heat transport on various parameters in the equations. It is found from the numerical calculations that: (1) The anomalous heat flux obtained from the bifurcation an..ysis in Section III is in good agreement with the one obtained from the numerical simulations near the linear stability limit D_L; (2) The anomalous heat flux varies weakly with the normalized diffusion coefficients K_R and M_R; and (3) The Nusselt number Nu, which is the ratio of the total heat flux to the collisional heat flux, becomes therefore significantly large when the collisional heat conduction is small. In a realistic parameter range for fusion experiments such that $D = 0.2$ and $M_R, K_R = 1.0 \sim 10^{-2}$, the Nusselt number varies from 1 to 10, in other words, the steady

convection on each rational surface enhances the heat transport up to one order of the magnitude over the collisional heat conduction.

In this thesis, up to Section VI, we develop the theory which is applicable to both fully developed turbulence and some coherent motion such as steady convection. However, in Sections VII and VIII, where we actually estimate the anomalous heat transport, we only treat coherent motion of the plasma or the steady convection on each rational surface: interactions of modes on different rational surfaces are ignored and only the nonlinear interaction of the modes with the same helicity is considered. This assumption of the coherent structure of the modes significantly simplifies the physical model and allows a mathematically rigorous treatment of the system. However, for fully developed turbulence, the anomalous heat transport can have some different features. Recently Carreras et al.[23] have studied the fully developed turbulence caused by the resistive g-mode, based on a set of reduced equations similar to the system (T). They estimate the anomalous heat transport analytically by using renormalization techniques of turbulence, and numerically by solving the reduced equations in the entire plasma so that they are able to take into account interactions of modes on different rational surfaces. It appears that their estimate of the anomalous heat transport of fully developed turbulence is about one order of magnitude larger than our result for the steady convection. Thus our results in Sections VII and VIII give good estimates of the anomalous heat in a relatively quiet plasma, where we may expect the steady convection to persist, not to develop into turbulence.

APPENDIX A

REDUCED EQUATIONS WITH PARALLEL VISCOSITY

Taking into account the effect of the parallel viscosity $\mu_{||}$ as well as the perpendicular viscosity $\mu_{\perp}$ and the heat conductivity κ, we will derive the set of nonlinear reduced equations of the resistive g-mode fluctuations. In addition to the orderings introduced in Section III, we assume that $\mu_{||} = O(1)$. We also assume that $\nabla_{\perp} \cdot \underset{\sim}{v}_1 = O(\delta)$ as in Section III so that

$$\lambda_1 = \frac{1}{\underset{\sim}{B}_o^2} \underset{\sim}{B}_o \cdot [(\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp}) \cdot \nabla \underset{\sim}{v}_1] - \frac{1}{3} \nabla_{\perp} \cdot \underset{\sim}{v}_1 , \tag{A-1}$$

which is derived from Eq. (2-1g), is $O(\delta)$ and the (i,j) component of the rate of strain tensor $\underset{\approx}{\sigma}$, i.e.,

$$(\underset{\approx}{\sigma}_1)_{ij} = \frac{\partial v_{1i}}{\partial x_j} + \frac{\partial v_{1j}}{\partial x_i} \tag{A-2}$$

is $O(1)$. These orderings are consistent with the assumptions made in Section II.

Corresponding to Eq. (3-1), the momentum equation of $\underset{\sim}{v}_1$ is given, up to $O(\delta)$, by

$$\rho_o(\frac{\partial}{\partial t} + \underset{\sim}{v}_1 \cdot \nabla)\underset{\sim}{v}_1$$

$$= -\nabla(p_1 + \underset{\sim}{B}_o \cdot \underset{\sim}{B}_1) + (\underset{\sim}{B}_o \cdot \nabla)\underset{\sim}{B}_1 + (\underset{\sim}{B}_1 \cdot \nabla)\underset{\sim}{B}_o + (\underset{\sim}{B}_1 \cdot \nabla)\underset{\sim}{b}_1 - (\nabla \cdot \underset{\approx}{\Pi})_1 . \tag{A-3}$$

Here

$$(\nabla\cdot\underset{\approx}{\Pi})_1 = -3\mu_{||}\,\underset{\sim}{B}_o(\underset{\sim}{B}_o\cdot\underset{\sim}{B}_{1_\perp})\cdot\nabla\left(\frac{\lambda_1}{B_o^2}\right) + \mu_{||}\nabla\lambda_1 - \mu_\perp\left(\Delta\underset{\sim}{v}_1 + \frac{1}{3}\nabla(\nabla\cdot\underset{\sim}{v}_1)\right) .$$

Therefore, the perpendicular component of Eq. (A-3) becomes

$$\nabla_\perp(p_1 + \underset{\sim}{B}_o\cdot\underset{\sim}{B}_1 - 3\mu_{||}\lambda_1) = 0 \tag{A-4}$$

to $O(1)$. Since $\nabla_\perp\cdot\underset{\sim}{B}_1 = 0$ holds to $O(1)$, we are able to introduce two scalar functions A and $b_{||}$ such that

$$\underset{\sim}{b}_1 = \nabla_\perp A \times \underset{\sim}{b} - b_{||}\underset{\sim}{b} \tag{A-5}$$

as in Section III. From Eq. (A-4), we write the parallel component of the fluctuating magnetic field, i.e., $b_{||} = -\underset{\sim}{B}_o\cdot\underset{\sim}{B}_1$ as

$$b_{||} = p_1 + 3\mu_{||}\lambda_1 , \tag{A-6}$$

assuming the mode vanishes away from the rational surface. With the scalar functions ϕ and ν introduced in Eq. (3-6), the operation $\underset{\sim}{B}_o\cdot\nabla\times$ (Eq. (A-3)) yields

$$\rho_o\frac{d}{dt}\Delta_\perp\phi = (\underset{\sim}{B}_o+\underset{\sim}{B}_1)\cdot\nabla(\Delta_\perp A) - 2\underset{\sim}{b}\times\nabla\left(p_o + \frac{1}{2}\underset{\sim}{B}_o^2\right)\cdot\nabla b_{||}$$

$$-2\underset{\sim}{b}\cdot\nabla\left(\frac{1}{2}\underset{\sim}{B}_o^2\right)(\Delta_\perp A) - 3\mu_{||}(\underset{\sim}{J}_o\cdot\underset{\sim}{B}_o)(\underset{\sim}{B}_o\cdot\underset{\sim}{B}_{1_\perp})\cdot\nabla\left(\frac{\lambda_1}{\underset{\sim}{B}_o^2}\right) + \mu_\perp\Delta_\perp^2\phi , \tag{A-7}$$

which corresponds to Eq. (3-9). The parallel component of Eq. (A-3) becomes

$$\rho_o \frac{d}{dt} \nu = \underset{\sim}{B}_o \cdot \nabla p_1 + \underset{\sim}{B}_{1\perp} \cdot \nabla b_{||} + \underset{\sim}{B}_1 \cdot \nabla p_o - 3\mu_{||} \underset{\sim}{B}_o^2 (\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp}) \cdot \nabla \left(\frac{\lambda_1}{B_o^2}\right)$$

$$+ \mu_{||} \underset{\sim}{B}_o \cdot \nabla \lambda_1 + \mu_\perp \Delta_\perp \nu \qquad \text{(A-8)}$$

The variable p_1 in Eq. (3-8) should be replaced by $b_{||}$ in our notation:

$$\frac{d}{dt} b_{||} - (\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp}) \cdot \nabla \nu - \underset{\sim}{v}_{1\perp} \cdot \nabla (p_o + \underset{\sim}{B}_o^2) - \underset{\sim}{B}_o^2 (\nabla_\perp \cdot \underset{\sim}{v}_1) - \eta \Delta_\perp b_{||} = 0 \,. \qquad \text{(A-9)}$$

The effect of viscosity does not enter the pressure equation to its lowest order under the assumption of our orderings. Therefore we may use Eq. (3-4) as the pressure equation here.

Now we express λ_1 in a more convenient form. From Eq. (A-1), we easily obtain

$$\lambda_1 = - \frac{1}{\underset{\sim}{B}_o^2} \left((\underset{\sim}{B}_o + \underset{\sim}{B}_{1\perp}) \cdot \nabla \nu + \underset{\sim}{v}_1 \cdot \nabla \left(p_o + \frac{1}{2} \underset{\sim}{B}_o^2\right) \right) - \frac{1}{3} \nabla_\perp \cdot \underset{\sim}{v}_1 \,. \qquad \text{(A-10)}$$

Therefore Eqs. (A-6)-(A-10), along with Eqs. (3-3), (3-4), (3-7) and (3-11), form the closed set of equations for the resistive g-mode fluctuations.

We now derive the nonlinear reduced equations of fluctuations in terms of the scaled variables, as in Section III. Let us assume that the plasma is confined in a cylinder and all mean quantities depend

only on its radius r, as before. In addition to the scaled variables (3-16), (3-20), and the parameters (3-17), (3-19), we use the following scaled variables

$$\tilde{\lambda} = -\frac{2r_o\sqrt{\rho_o}}{B_\theta|\sigma|^{5/2}}\lambda_1 , \qquad \tilde{p} = \frac{2}{\rho_o|\sigma|^{3/2}}\rho_1 ,$$

$$\tilde{b} = \frac{2}{B^2|\sigma|^{3/2}} b_{||} , \qquad \tilde{\alpha} = \frac{2r_o\sqrt{\rho_o}}{B_\theta|\sigma|^{5/2}}(\nabla_\perp \cdot \underset{\sim}{v}_1) ,$$

and the following parameters

$$N = \frac{-r_o\rho_o'}{\rho_o|\sigma|^2} , \qquad H = \frac{-2r_o(p_o+B^2)'}{B^2|\sigma|^2} ,$$

$$M_\perp = \frac{\mu_\perp}{r_oB_\theta\sqrt{\rho_o}} , \qquad M_{||} = \frac{3\mu_{||}B_\theta|\sigma|}{r_oB^2\sqrt{\rho_o}} ,$$

$$J = \frac{1}{2}\frac{r_o\sigma}{B_\theta B|\sigma|^{1/2}}(\underset{\sim}{J}_o \cdot \underset{\sim}{B}_o) .$$

Here we note that the parameters N and H are not independent of the parameters defined in Eqs. (3-19):

$$H = S - D$$

$$D = \frac{\gamma-1}{\gamma}\theta + \frac{\beta}{2}N .$$

It follows that Eqs. (A-6) and (3-11) become

$$\tilde{b} = \tilde{p} - \tilde{M}_{||}\tilde{\lambda} \qquad \text{(A-11)}$$

and

$$\frac{\beta}{2}\tilde{\rho} = \tilde{p} - \frac{\gamma-1}{\gamma}\tilde{T} , \qquad \text{(A-12)}$$

respectively. From Eq. (A-10), we have

$$\tilde{\lambda} = \frac{\partial\tilde{v}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{v}\} + \frac{S}{2}\frac{\partial\tilde{\phi}}{\partial y} + \frac{1}{3}\tilde{\alpha} . \qquad \text{(A-13)}$$

Rewriting Eqs. (A-6)-(A-12), (3-3), (3-4), (3-7) and (3-11) in terms of the scaled variable, we obtain the following set of equations, which, together with Eqs. (A-11)-(A-13), describe the resistive g-mode fluctuations with the parallel viscosity:

$$\frac{d\tilde{A}}{d\tau} = \frac{\partial\tilde{\phi}}{\partial\tilde{\theta}} + \frac{1}{R}\tilde{\Delta}_{\perp}\tilde{A}$$

$$\frac{d}{d\tau}\tilde{\Delta}_{\perp}\tilde{\phi} = \frac{\partial}{\partial\tilde{\theta}}\tilde{\Delta}_{\perp}\tilde{A} + \{\tilde{A},\tilde{\Delta}_{\perp}\tilde{A}\} - \frac{\partial\tilde{b}}{\partial y} + JM_{||}\left(\frac{\partial\tilde{\lambda}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{\lambda}\}\right) + M_{\perp}\tilde{\Delta}^{2}_{\perp}\tilde{\phi}$$

$$\frac{d}{d\tau}\tilde{\rho} = -N\frac{\partial\tilde{\phi}}{\partial y} - \tilde{\alpha}$$

$$\frac{d}{d\tau}\tilde{b} = H\frac{\partial\tilde{\phi}}{\partial y} + \frac{\partial\tilde{v}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{v}\} + \tilde{\alpha} + \frac{1}{R}\tilde{\Delta}_{\perp}\tilde{b}$$

$$\frac{d}{d\tau}\tilde{p} = -D\frac{\partial\tilde{\phi}}{\partial y} - \frac{\gamma\beta\tilde{\alpha}}{2} + (\gamma-1)(K_{||}\tilde{\Delta}_{||} + K_{\perp}\tilde{\Delta}_{\perp})\tilde{T}$$

$$\frac{d\tilde{v}}{d\tau} = \frac{\partial\tilde{p}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{b}_{||}\} + D\frac{\partial\tilde{A}}{\partial y} + M_{||}\left(\frac{2}{3}\frac{\partial\tilde{\lambda}}{\partial\tilde{\theta}} + \frac{1}{3}\{\tilde{A},\tilde{\lambda}\}\right) + M_{\perp}\tilde{\Delta}_{\perp}\tilde{v} .$$

These equations are the extension of Eqs. (R) given in Section III.

APPENDIX B

THE BOUNDARY CONDITIONS AND THE UNIQUENESS OF THE SOLUTIONS

In this Appendix, we will give a brief discussion of the boundary conditions introduced in Section IV and we will prove the uniqueness of the solutions of the system (T) with these boundary conditions, assuming the existence of the solutions. The objective of this Appendix is to show that the boundary conditions are the natural ones for the nonlinear reduced equations (R) and (T) although we do not intend to prove the well-posedness of the initial-boundary value problems of these systems.

We consider the solutions of the system (R) [or (T)] satisfying the following conditions: $\tilde{A} \in C^3(\bar{\Omega})$, $\tilde{\phi} \in C^4(\bar{\Omega})$, $\tilde{p}, \tilde{T}, \tilde{\nu} \in C^2(\bar{\Omega})$, where Ω is the bounded doman defined by

$$\Omega = \{(\tilde{x},\tilde{y},\tilde{z}) \in R^3 : |\tilde{x}| < \delta_x, \quad |\tilde{y}| < \delta_y , \quad 0 < \tilde{z} < L_z\} . \quad \text{(B-1)}$$

It is easy to check that the boundary conditions (I), (II) (or II') and (III) are necessary and sufficient to derive "the energy equations" (4-6) and (4-7) from the system (R). [For the system (T), those boundary conditions for $\tilde{\phi}$, $\tilde{\Delta}_\perp\tilde{\phi}$, $\tilde{A}$ and $\tilde{p}$ are necessary and sufficient to derive Eqs. (4-6) and (5-3)]. For example, we need to specify the boundary values of $\tilde{\Delta}_\perp\tilde{\phi}$ as well as $\tilde{\phi}$ in order to eliminate the surface integral from the following integral (Green's identity)

$$\int_{\Omega^2} \tilde{\phi}\tilde{\Delta}_\perp^2\tilde{\phi}\, d\tilde{x}d\tilde{y}$$

$$= \int_{\Omega^2} |\tilde{\nabla}_\perp\tilde{\phi}|^2\, d\tilde{x}d\tilde{y} + \int_{\partial\Omega^2} (\tilde{\phi}\, \frac{d\tilde{\Delta}_\perp\tilde{\phi}}{dn} - \tilde{\Delta}_\perp\tilde{\phi}\, \frac{d\tilde{\Delta}_\perp\tilde{\phi}}{dn})dS\ ,$$

where

$$\Omega^2 = \{(\tilde{x},\tilde{y}) \in R^2 : |\tilde{x}| < \delta_x\ ,\quad |\tilde{y}| < \delta_y\}, \tag{B-2}$$

and d/dn denotes differentiation in the direction of the exterior normal to the boundary $\partial\Omega^2$. This integration is used to derive Eq. (4-6).

We now prove the uniqueness of the solutions of the system (T). Suppose that there exist solutions $\tilde{A}_1,\tilde{A}_2 \in C^2(\bar{\Omega})$, $\tilde{\phi}_1,\tilde{\phi}_2 \in C^4(\bar{\Omega})$ and $\tilde{p}_1,\tilde{p}_2 \in C^2(\bar{\Omega})$, satisfying the boundary conditions (I), (II) [or (II')] and (III) from $\tau = 0$ to $\tau = T_f$, such that $\tilde{A}_1 = \tilde{A}_2$, $\tilde{\phi}_1 = \tilde{\phi}_2$ and $\tilde{p}_1 = \tilde{p}_2$ at $\tau = 0$. Let $A^* = \tilde{A}_2-\tilde{A}_1$, $\phi^* = \tilde{\phi}_2-\tilde{\phi}_1$ and $p^* = \tilde{p}_2-\tilde{p}_1$, then these variables satisfy the following equation:

$$\frac{d}{d\tau}\, \varepsilon^*(\tau) = -\langle\phi^*\{\tilde{\phi}_1,\tilde{\Delta}_\perp\phi^*\}\rangle - \langle p^*\{\phi^*,\tilde{p}_1\}\rangle - (1+D)\langle p^*\, \frac{\partial\phi^*}{\partial\tilde{y}}\rangle - \underline{F}^*(\tau)\ , \tag{B-3}$$

where

$$\varepsilon^*(\tau) = \frac{1}{2}\, (\langle|\tilde{\nabla}_\perp A^*|^2\rangle + \langle|\tilde{\nabla}_\perp\phi^*|^2\rangle + \langle|p^*|^2\rangle),$$

and

$$\underline{F}^*(\tau) = \frac{1}{R}\, \langle|\tilde{\Delta}_\perp A^*|^2\rangle + M\langle|\tilde{\Delta}_\perp\phi^*|^2\rangle + \chi\langle|\tilde{\nabla}_\perp p^*|^2\rangle\ .$$

Equation (B-3) may be derived in the way similar to deriving Eq. (6-6) from Eqs. (T). We note that $\varepsilon^*(0) = 0$ from the initial conditions.

Using Schwartz's inequality, we obtain

$$|\langle p^* \frac{\partial\phi^*}{\partial\tilde{y}}\rangle| \leq (\langle|p^*|^2\rangle\langle|\tilde{\nabla}_\perp\phi^*|^2\rangle)^{1/2}$$

$$\leq \frac{1}{2}(\langle|p^*|^2\rangle + \langle|\tilde{\nabla}_\perp\phi^*|^2\rangle)$$

$$\leq \varepsilon^*(\tau) \quad . \tag{B-4}$$

Similarly

$$|\langle p^* \frac{\partial\phi^*}{\partial\tilde{x}} \frac{\partial\tilde{p}_1}{\partial\tilde{y}}\rangle| \leq \langle|p^* \frac{\partial\phi^*}{\partial\tilde{x}}|\rangle \, ||\frac{\partial\tilde{p}_1}{\partial\tilde{y}}||_\infty$$

$$\leq ||\frac{\partial\tilde{p}_1}{\partial\tilde{y}}||_\infty \, \varepsilon^*(\tau) \quad , \tag{B-5}$$

where $||f||_\infty = \sup_{\tilde{x} \in \Omega} |f(\tilde{x})|$ for $f \in C(\bar{\Omega})$. Since $\tilde{p}_1 \in C^2(\bar{\Omega})$, there exists a positive constant C_1 such that

$$|\langle p^*\{\phi^*,\tilde{p}_1\}\rangle| \leq C_1 \, \varepsilon^*(\tau) \quad . \tag{B-6}$$

Taking into account the following identity

$$\{\tilde{\phi}_1, \tilde{\Delta}_\perp \phi^*\} = \frac{\partial}{\partial \tilde{x}} \{\tilde{\phi}_1, \frac{\partial \phi^*}{\partial \tilde{x}}\} + \frac{\partial}{\partial \tilde{y}} \{\tilde{\phi}_1, \frac{\partial \phi^*}{\partial \tilde{y}}\} - \{\frac{\partial \tilde{\phi}_1}{\partial \tilde{x}}, \frac{\partial \phi^*}{\partial \tilde{x}}\} - \{\frac{\partial \tilde{\phi}_1}{\partial \tilde{y}}, \frac{\partial \phi^*}{\partial \tilde{y}}\} ,$$

we obtain

$$\langle \phi^* \{\tilde{\phi}_1, \tilde{\Delta}_\perp \phi^*\} \rangle = \langle \frac{\partial \phi^*}{\partial \tilde{x}} \{\frac{\partial \tilde{\phi}_1}{\partial \tilde{x}}, \phi^*\} + \frac{\partial \phi^*}{\partial \tilde{y}} \{\frac{\partial \tilde{\phi}_1}{\partial \tilde{y}}, \phi^*\} \rangle .$$

Therefore, there exist constants C_2 and $\tilde{C}_2$ such that

$$\begin{aligned} |\langle \phi^* \{\tilde{\phi}_1, \tilde{\Delta}_\perp \phi^*\} \rangle| &\le \tilde{C}_2 \langle |\tilde{\nabla}_\perp \phi^*|^2 \rangle \\ &\le C_2 \, \varepsilon^*(\tau) , \end{aligned} \tag{B-7}$$

the derivation of which is similar to the derivation of Eqs. (B-5) and (B-6).

Thus, from Eqs. (6-10), (B-3), (B-4), (B-6) and (B-7), we have

$$\begin{aligned} \frac{d}{d\tau} \varepsilon^*(\tau) &\le \left(C_2 + C_1 + |1+D| - \frac{\Lambda}{C}\right) \varepsilon^*(\tau) \\ &= C_3 \, \varepsilon^*(\tau) , \end{aligned} \tag{B-8}$$

where C_3 is a constant (which is not necessarily positive). It follows that, since $\varepsilon^*(0) = 0$,

$$0 \geq \int_0^\tau \left(\frac{d\varepsilon^*(t)}{dt} - C_3\, \varepsilon^*(t)\right) \exp\,(-C_3 t)dt$$

$$= \int_0^\tau \frac{d}{dt}\left[\varepsilon^*(t)\, \exp\,(-C_3 t)\right]dt$$

$$= \varepsilon^*(\tau)\, \exp\,(-C_3 \tau)\ ,$$

for all $\tau \leq T_f$. Therefore, we have $\varepsilon^*(\tau) = 0$, i.e., $A^* = \phi^* = p^* = 0$, for all $\tau \leq T_f$, which proves the uniquenss of the solutions of the system (T).

To conclude, we have shown that the boundary conditions introduced in Section IV are necessary and sufficient to derive all the results in this thesis and to guarantee the uniqueness of the solutions of the system (T). We also note that these boundary conditions are used to solve the reduced equations numerically (Section VIII). These facts suggest that the boundary conditions imposed on the systems (R) and (T) are the natural ones although we did not prove the well-posedness of the problem.

APPENDIX C

ON THE MAXIMUM OF $-\langle \tilde{p}\partial\tilde{\phi}/\partial\tilde{y}\rangle/\underline{F}$

In this Appendix, we will briefly discuss the existence of the functions $\tilde{A}_o$, $\tilde{\phi}_o$ and $\tilde{p}_o$ that maximize the functional $-\langle \tilde{p}\partial\tilde{\phi}/\partial\tilde{y}\rangle/\underline{F}(\tau)$ (Eq. (6-7)). The existence of these functions is used to prove the corollary of Theorem 1 in Section VI. For this purpose, we need to define the function space H introduced in the proof of Theorem 1 more specifically. The symbols, terminology and basic properties of function spaces to be used in this Appendix are found in Ref. 24.

In this thesis, as discussed in Appendix B, we have considered the solutions of the system (T) satisfying the following conditions: $\tilde{A} \in C^3(\bar{\Omega})$, $\tilde{\phi} \in C^4(\bar{\Omega})$ and $\tilde{p} \in C^2(\bar{\Omega})$, where Ω is defined by (B-1). However, since it is obvious that the maximum of $-\langle \tilde{p}\partial\tilde{\phi}/\partial\tilde{y}\rangle/\underline{F}$ is attained when $\langle|\tilde{\Delta}_\perp \tilde{A}|^2\rangle = 0$, i.e., $\tilde{A}_o \equiv 0$, because of the boundary conditions, we only need to take the supremum of the following functional of a pair of functions $u = (\tilde{\phi},\tilde{p})$:

$$\Psi(u) = \frac{-\langle \tilde{p}\,\dfrac{\partial\tilde{\phi}}{\partial\tilde{y}}\rangle}{M\langle|\tilde{\Delta}_\perp\tilde{\phi}|^2\rangle + \chi\langle|\tilde{\nabla}_\perp\tilde{p}|^2\rangle} .$$

Moreover, since Ψ does not involve any $\tilde{z}$-derivatives of $\tilde{\phi}$ and $\tilde{p}$, we may think of $\tilde{\phi}$ and $\tilde{p}$ as functions of only $\tilde{x}$ and $\tilde{y}$ without loss of generality. Therefore, we define the function space H as

$$H = \{(\tilde{\phi},\tilde{p}) : \tilde{\phi} \in C^4(\bar{\Omega}^2),\ \tilde{p} \in C^2(\bar{\Omega}^2),\ \text{and } \tilde{p},\tilde{\phi},\tilde{\Delta}_{\perp}\tilde{\phi} = 0 \ \text{ on } \partial\Omega^2\} , \qquad \text{(C-1)}$$

equipped with the norm

$$||(\tilde{\phi},\tilde{p})||_H = (||\tilde{\phi}||_4^2 + ||\tilde{p}||_2^2)^{1/2} . \qquad \text{(C-2)}$$

Here Ω^2 is defined by (B-2) and

$$||f||_k = (\sum_{0 \leq |\alpha| \leq k} ||D^\alpha f||^2)^{1/2}$$

for $f \in C^k(\bar{\Omega})$, where $||\cdot||$ is the L^2-norm and $D^\alpha f$ is the (weak) partial derivative of f. We here imposed the boundary conditions (I) and (II) in Section IV: in order to use (II'), instead of (II), we need to change the boundary condition in (C-1) accordingly. With these definitions, the original maximum problem is reduced to

$$\sup_{\substack{u \in H \\ ||u||_H \neq 0}} \Psi(u) .$$

We now define the following two spaces:

$$\bar{H} = \{(\tilde{\phi},\tilde{p}) : \tilde{\phi} \in H^4(\Omega^2),\ \tilde{p} \in H^2(\Omega^2) \text{ and } \tilde{p},\tilde{\phi},\tilde{\Delta}_{\perp}\tilde{\phi} = 0 \ \text{ on } \partial\Omega^2\}$$

equipped with the norm $||\cdot||_H$ defined by Eq. (C-2), and

$$\tilde{H} = \{(\tilde{\phi},\tilde{p}) : \tilde{\phi} \in H^3(\Omega^2),\ \tilde{p} \in H^1(\Omega^2) \text{ and } \tilde{p},\tilde{\phi},\tilde{\Delta}_{\perp}\tilde{\phi} = 0 \ \text{ on } \partial\Omega^2\}$$

equipped with the norm

$$||(\tilde{\phi},\tilde{p})||_{\tilde{H}} = (||\tilde{\phi}||_3^2 + ||\tilde{p}||_1^2)^{1/2} .$$

Here $H^k(\Omega^2)$ is a Sobolev space over Ω^2. Clearly $\overline{H}$ is the completion of H with respect to the norm $||\cdot||_H$. We claim here that $(\tilde{\phi}_o, \tilde{p}_o) \in \tilde{H}$ or,

PROPOSITION C: There exists $u_o \in \tilde{H}$ such that

$$\Psi(u_o) = \sup_{\substack{u \in H \\ ||u||_H \neq 0}} \Psi(u) . \tag{C-3}$$

PROOF: As we have shown in the proof of Theorem 1, there exists the least upper bound of $|\Psi|$. Therefore there exists a sequence $\{u_n\}$ $(||u_n||_H \neq 0)$ in H such that $\Psi(u_n) \to$ the right-hand-side of Eq. (C-3) as $n \to \infty$. By forming a bounded sequence $\{u_n/||u_n||_H\}$, it is easy to check the right-hand-side of Eq. (C-3) is equal to

$$\sup_{\substack{u \in H \\ ||u||_H = 1}} \Psi(u) .$$

Hence the proposition follows from the fact that the imbedding $\overline{H} \to \tilde{H}$ is compact, which is an easy consequence of Rellich's theorem.

(Q.E.D.)

Thus we have defined the function space H introduced in Section VI more specifically and we have shown that the functions $(\tilde{\phi}_o, \tilde{p}_o)$ $\tilde{H}$ that maximize Ψ can be approximated by functions in H as closely as possible with respect to the norm $||\cdot||_{\tilde{H}}$. Assuming that there exist smooth enough solutions of the system (T) with these (less smooth) initial conditions $\tilde{\phi}_o$ and $\tilde{p}_o$ as well as $\tilde{A}_o \equiv 0$, the discussion in this Appendix completes the proof of the corollary of Theorem 1.

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FIGURE CAPTIONS

Fig. 1 A typical steady convection on a rational surface obtained from Eqs. (T) at time $\tau = 300\tau_A$. The profiles of the Fourier components of (a) $\tilde{A}/\delta^2$, (b) $\tilde{\Delta}_\perp\tilde{A}$, (c) $\tilde{\phi}/\delta^2$, (d) $\tilde{\Delta}_\perp\tilde{\phi}$, and (e) $\tilde{p}/\delta$ are shown as functions of $\tilde{x}/\delta$, where $\delta = \sqrt{R}$. Here 12 modes are included, $M_R = 1$, $K_R = 5\times10^{-3}$, and $D = 0.2$.

Fig. 2 Streamlines, i.e. contours of constant $\tilde{\phi}$, of convection cells, corresponding to Fig. 1(c).

Fig. 3 Time evolution of the Nusselt number Nu under the same condition as in Fig. 1.

Fig. 4 The Nusselt number Nu as a function of D. The straight line (a) is the analytical prediction from Eq. (7-20) and the curve (b) is from the simulation of Eqs. (T). Here $M_R = 1$ and $K_R = 10^{-2}$.

Fig. 5 The Nusselt number Nu as a function of D. The straight line (a) is the analytical prediction from Eq. (7-20) and the curve (b) is from the simulation of Eqs. (T). Here $M_R = 1$ and $K_R = 10^{-1}$.

Fig. 6 The dependence of the anomalous heat transport on the normalized heat conductivity K_R. The curve (a) is the Nusselt number Nu and the curve (b) is the anomalous heat flux $J_a = -p_o\langle s_1 v_{1r}\rangle/n\rho_o T_o'$. Here $M_R = 1$ and $D = 0.2$.

Fig. 7 The Nusselt number Nu as a function of the normalized viscosity M_R obtained from the simulation of Eqs. (T). The curve (a) is for $K_R = 10^{-2}$ and the curve (b) is for $K_R = 10^{-1}$. Here $D = 0.2$.

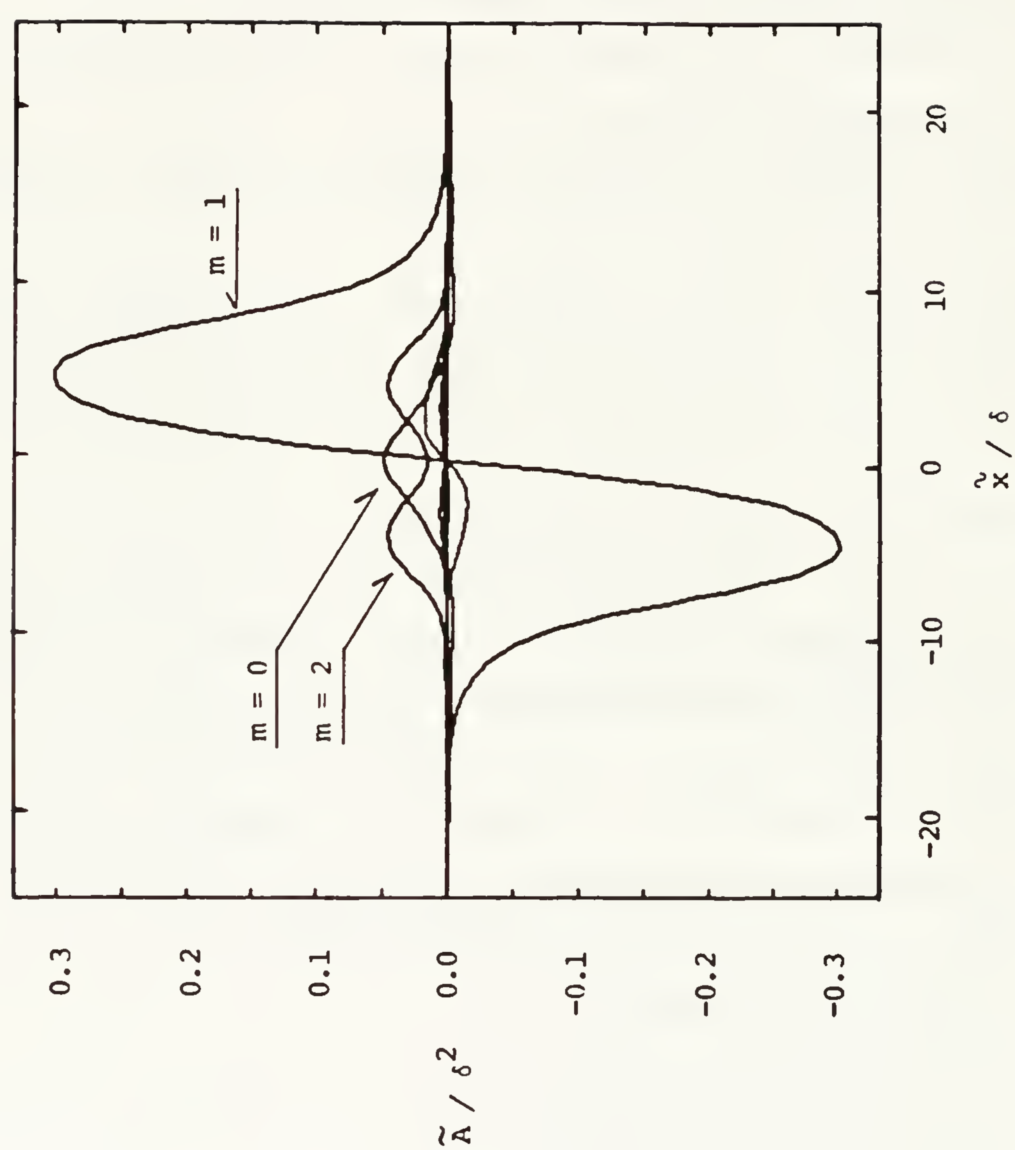
m = 1
m = 0
m = 2
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
$\tilde{A} / \delta^2$
-20
-10
0
10
20
$\tilde{x} / \delta$

Fig. 1 - (a)

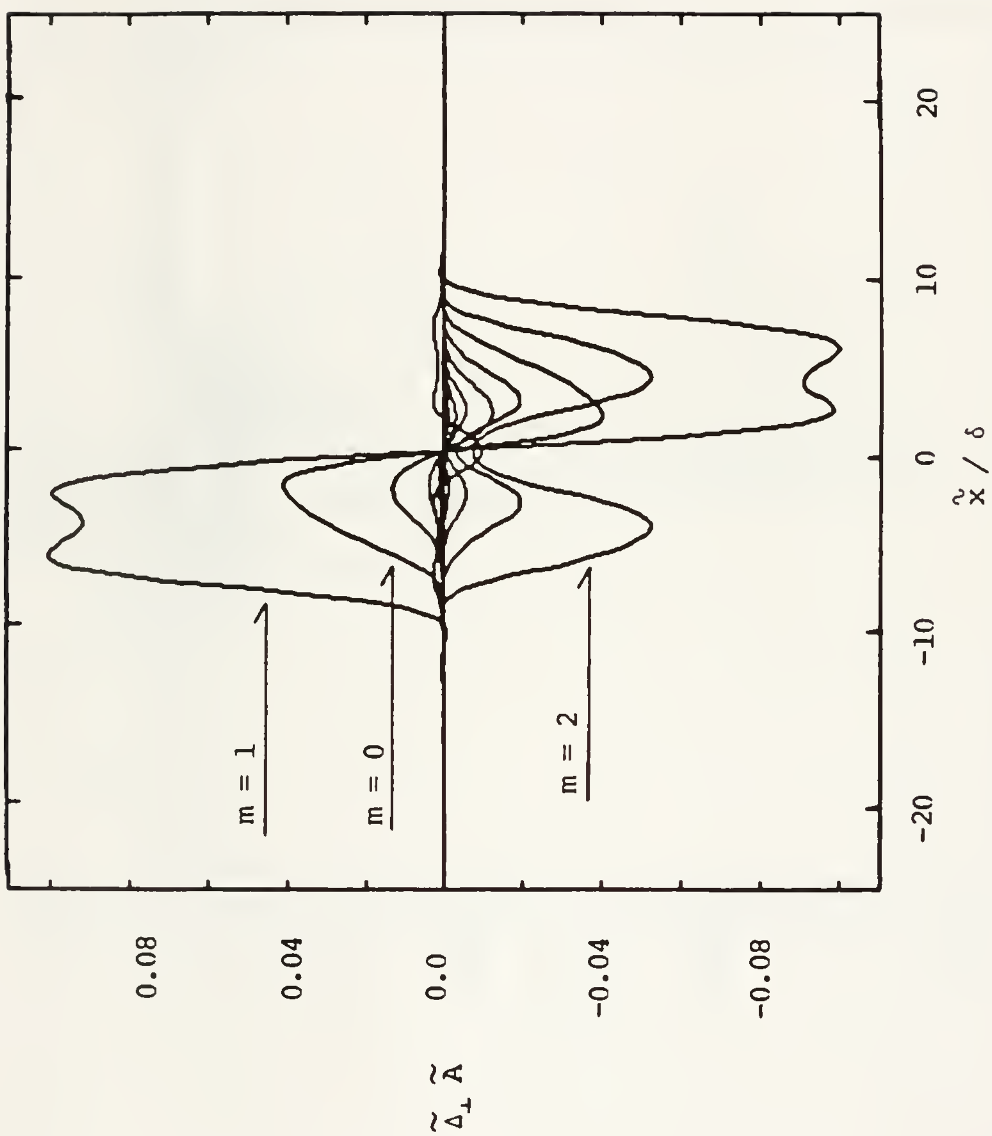
m = 1
m = 0
m = 2
0.08
0.04
0.0
-0.04
-0.08
$\tilde{\Delta}_\perp \tilde{A}$
-20
-10
0
10
20
$\tilde{x} / \delta$

Fig. 1 - (b)

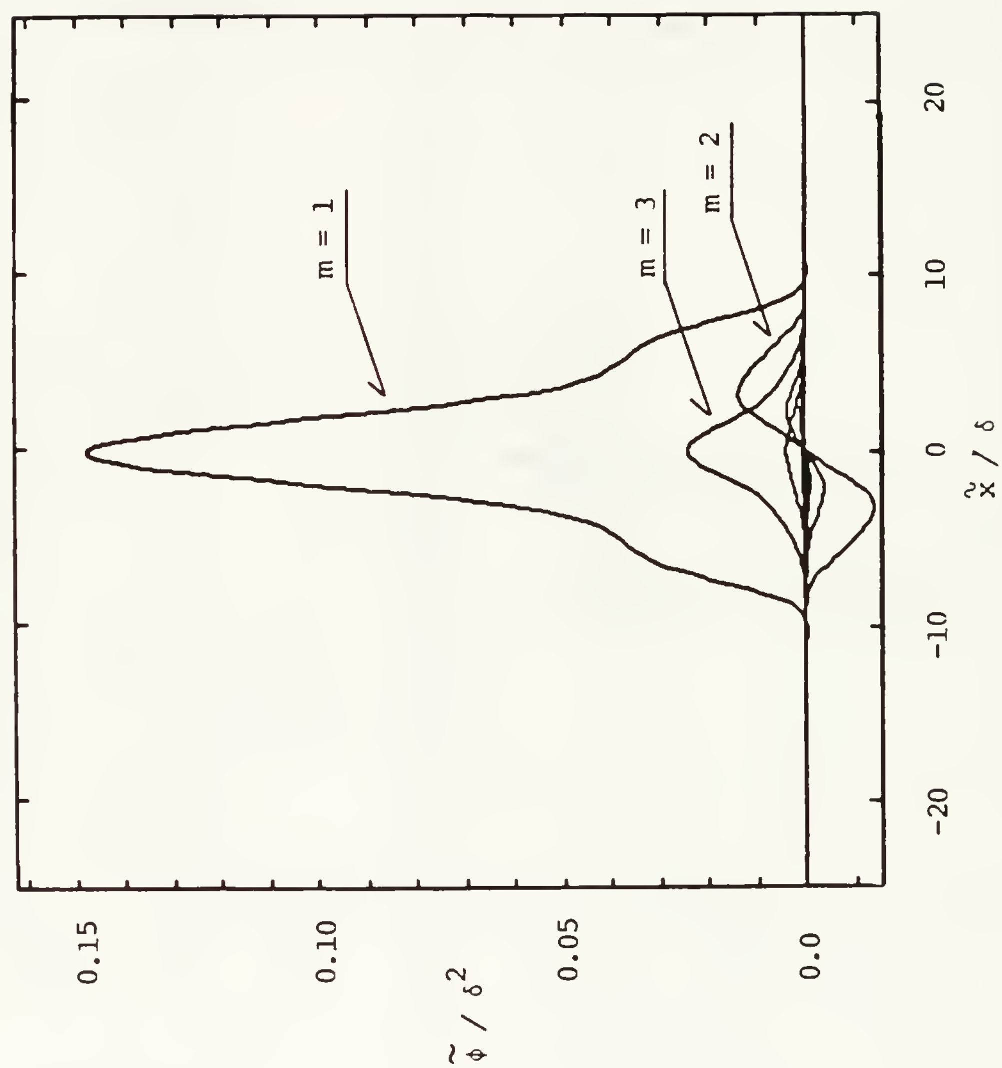
m = 1
m = 3
m = 2
0.15
0.10
0.05
0.0
$\tilde{\phi} / \delta^2$
-20
-10
0
10
20
$\tilde{x} / \delta$

Fig. 1 - (c)

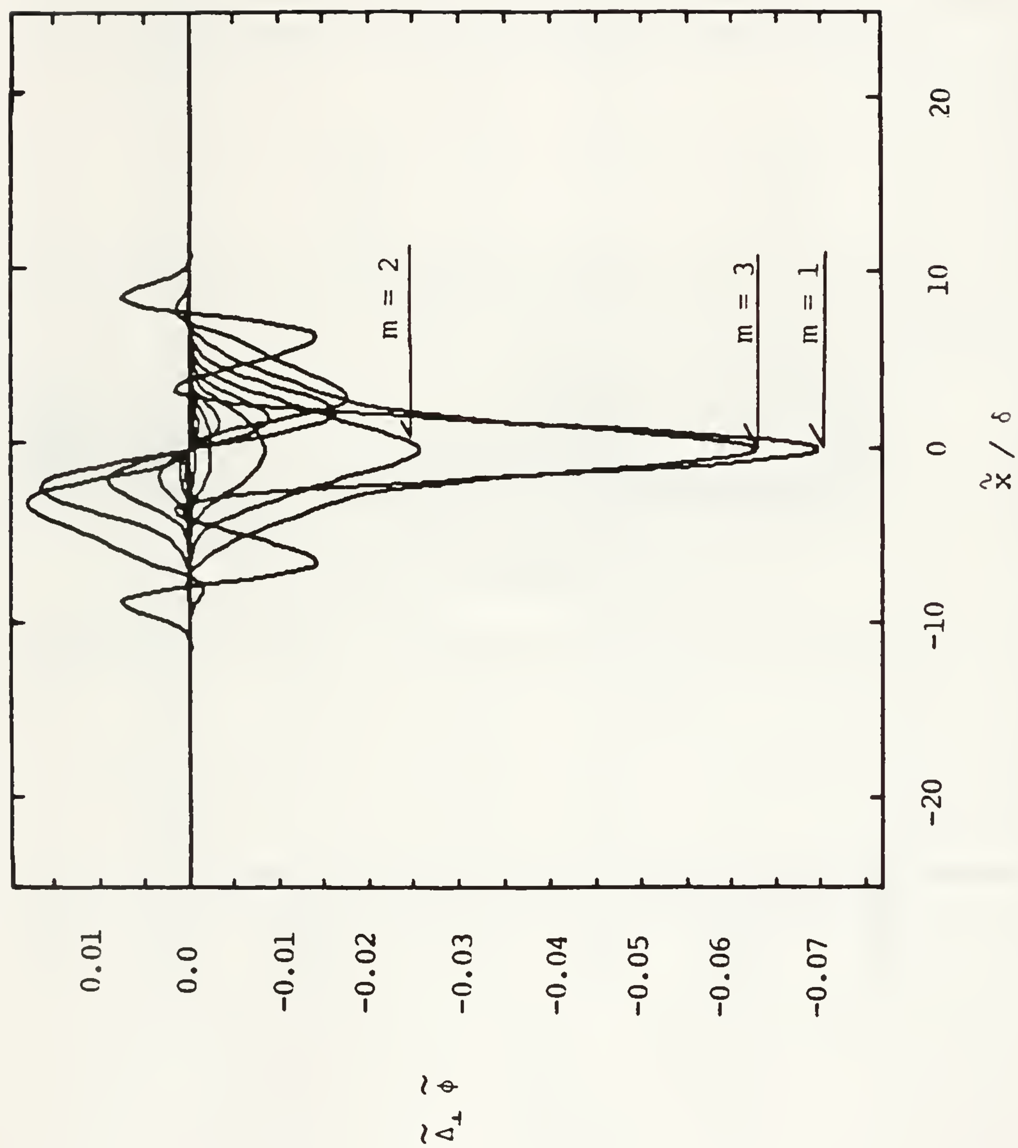
0.01
0.0
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
$\tilde{\Delta}_\perp \tilde{\phi}$
-20
-10
0
10
20
$\tilde{x} / \delta$
m = 2
m = 3
m = 1

Fig. 1 - (d)

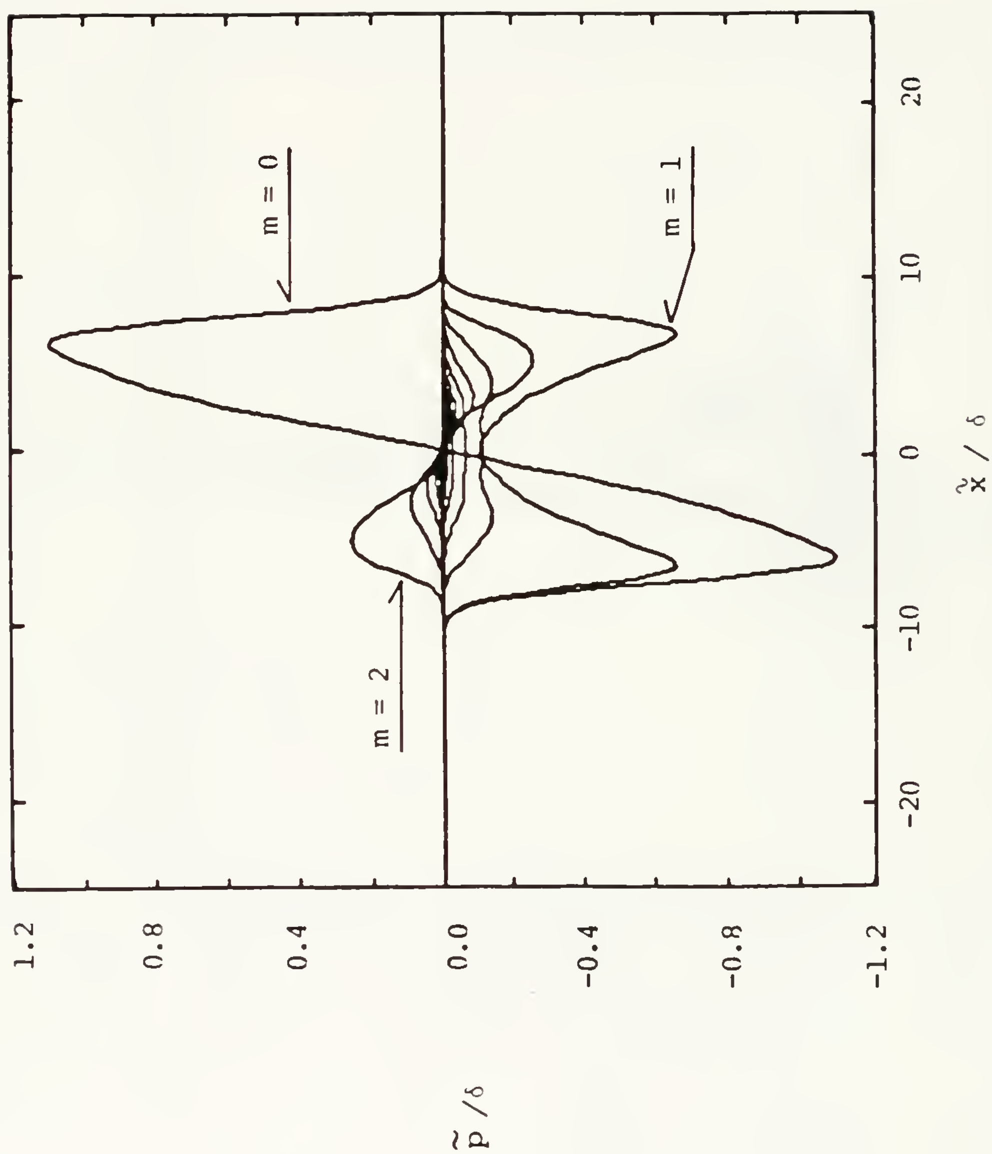
m = 0
m = 1
m = 2
1.2
0.8
0.4
0.0
-0.4
-0.8
-1.2
-20
-10
0
10
20
$\tilde{p}/\delta$
$\tilde{x}/\delta$

Fig. 1 - (e)

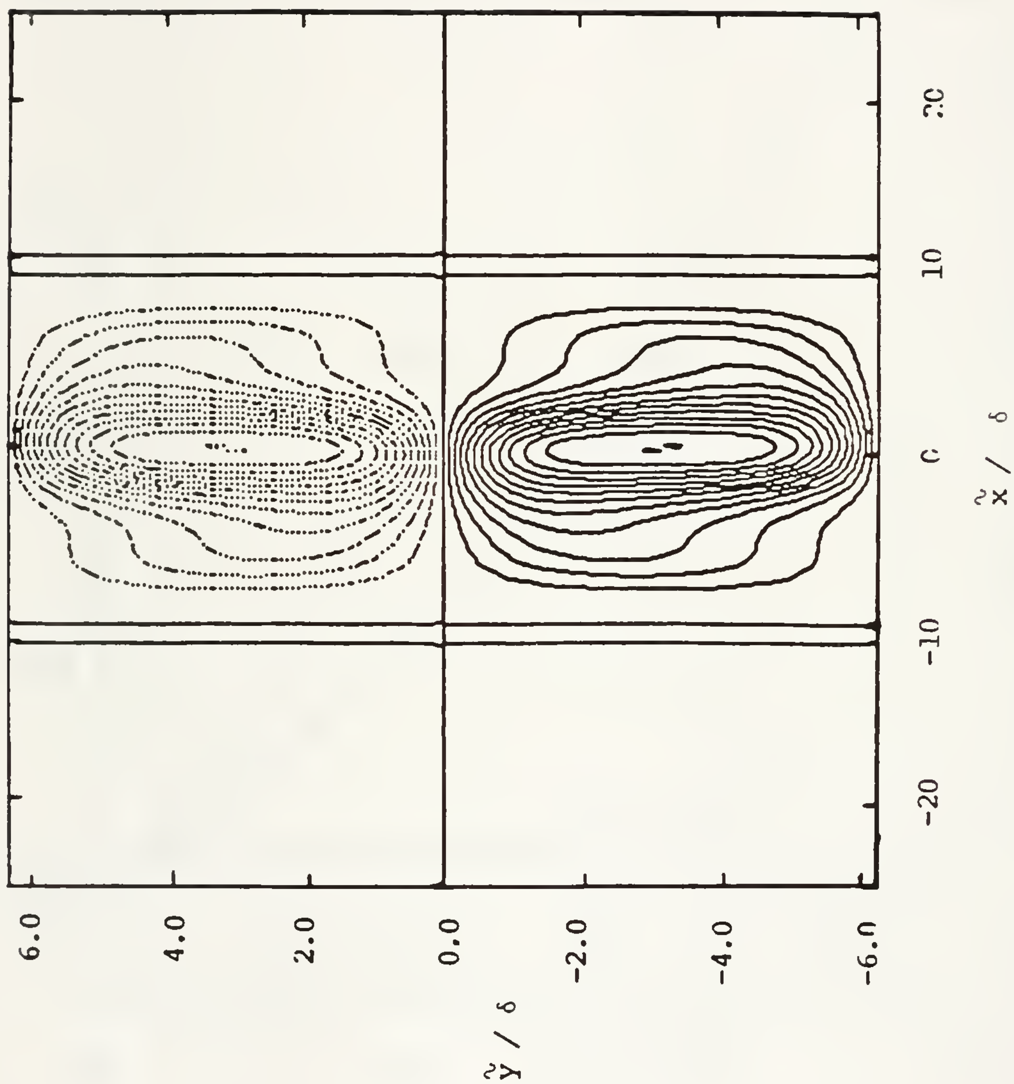

Fig. 2

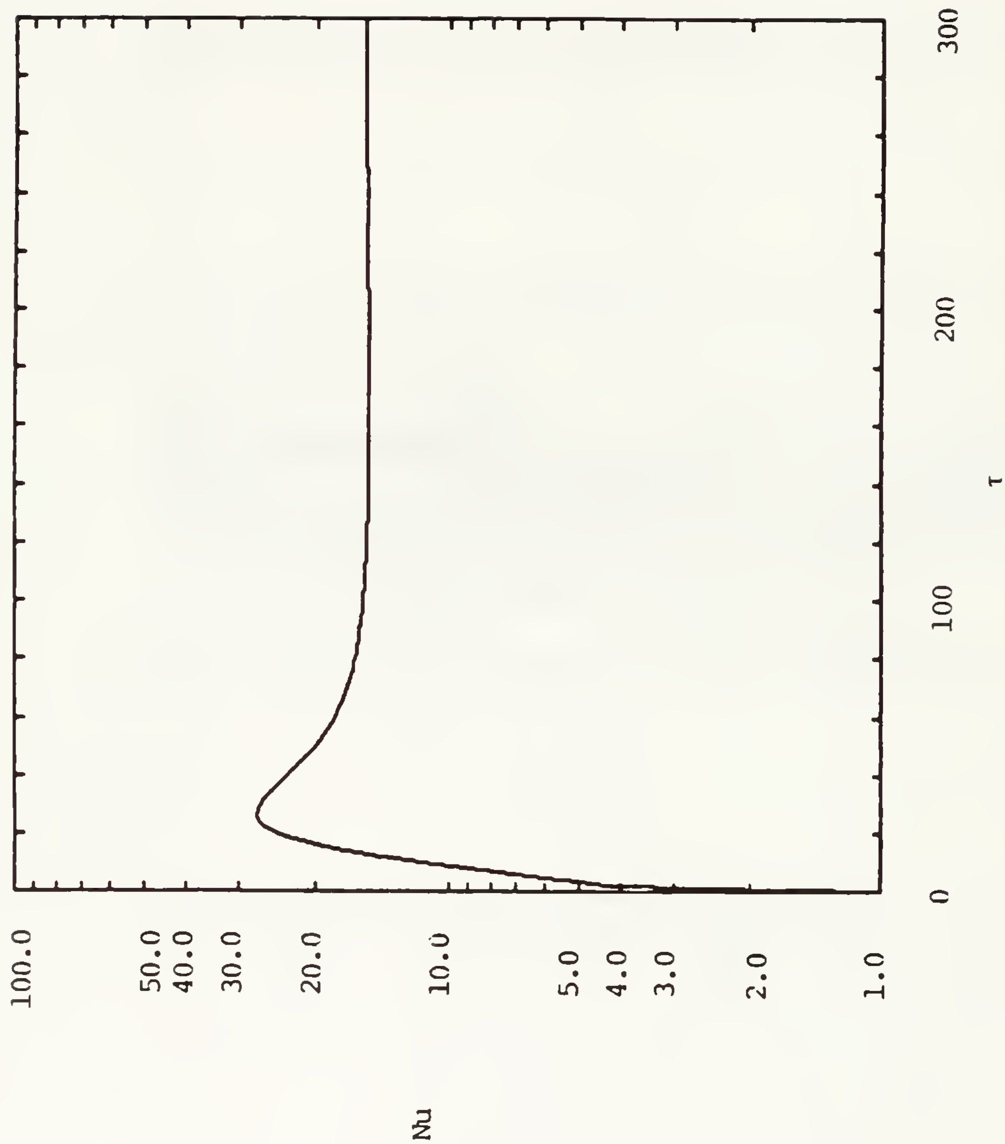

Fig. 3

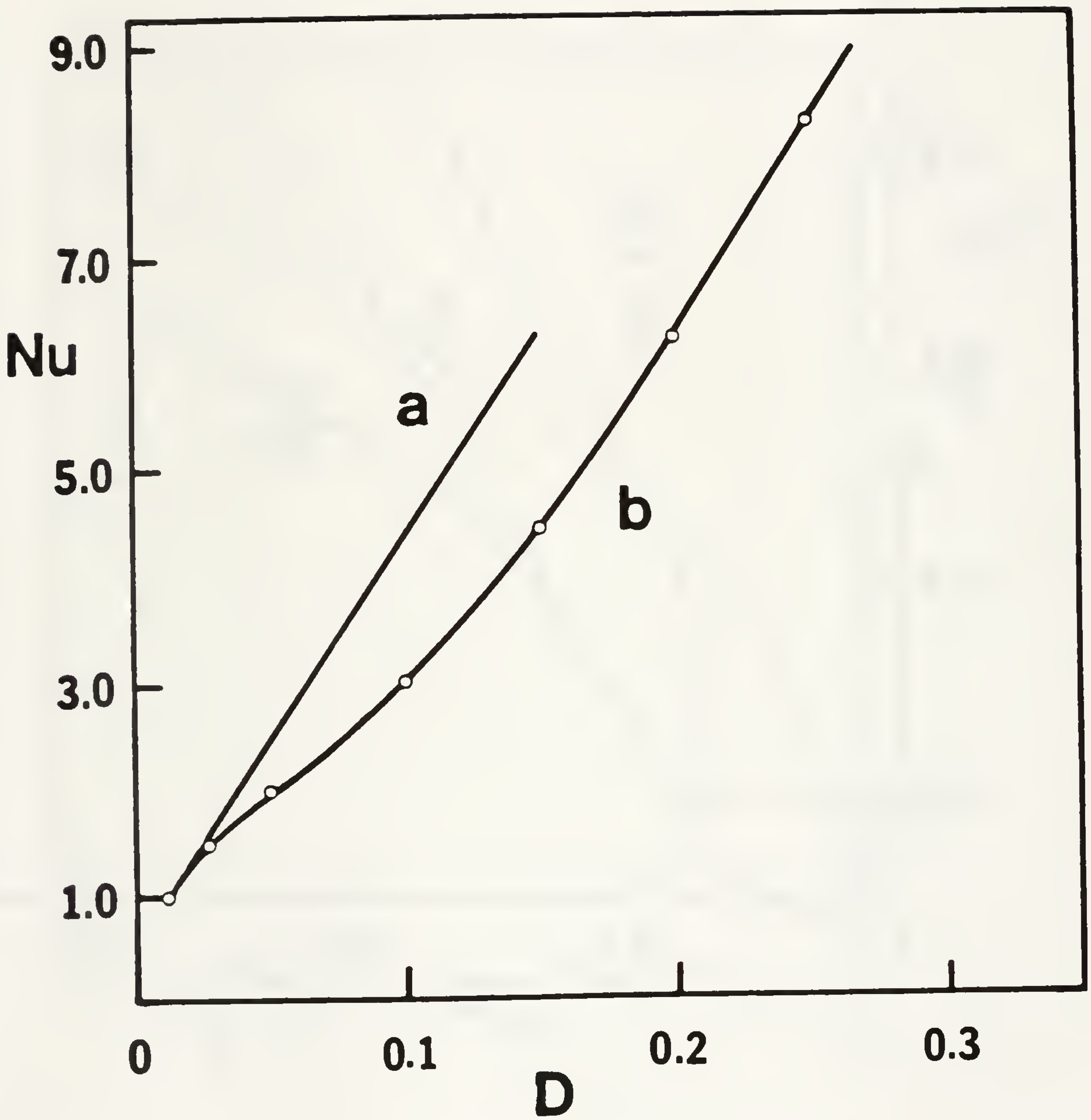

FIG 4

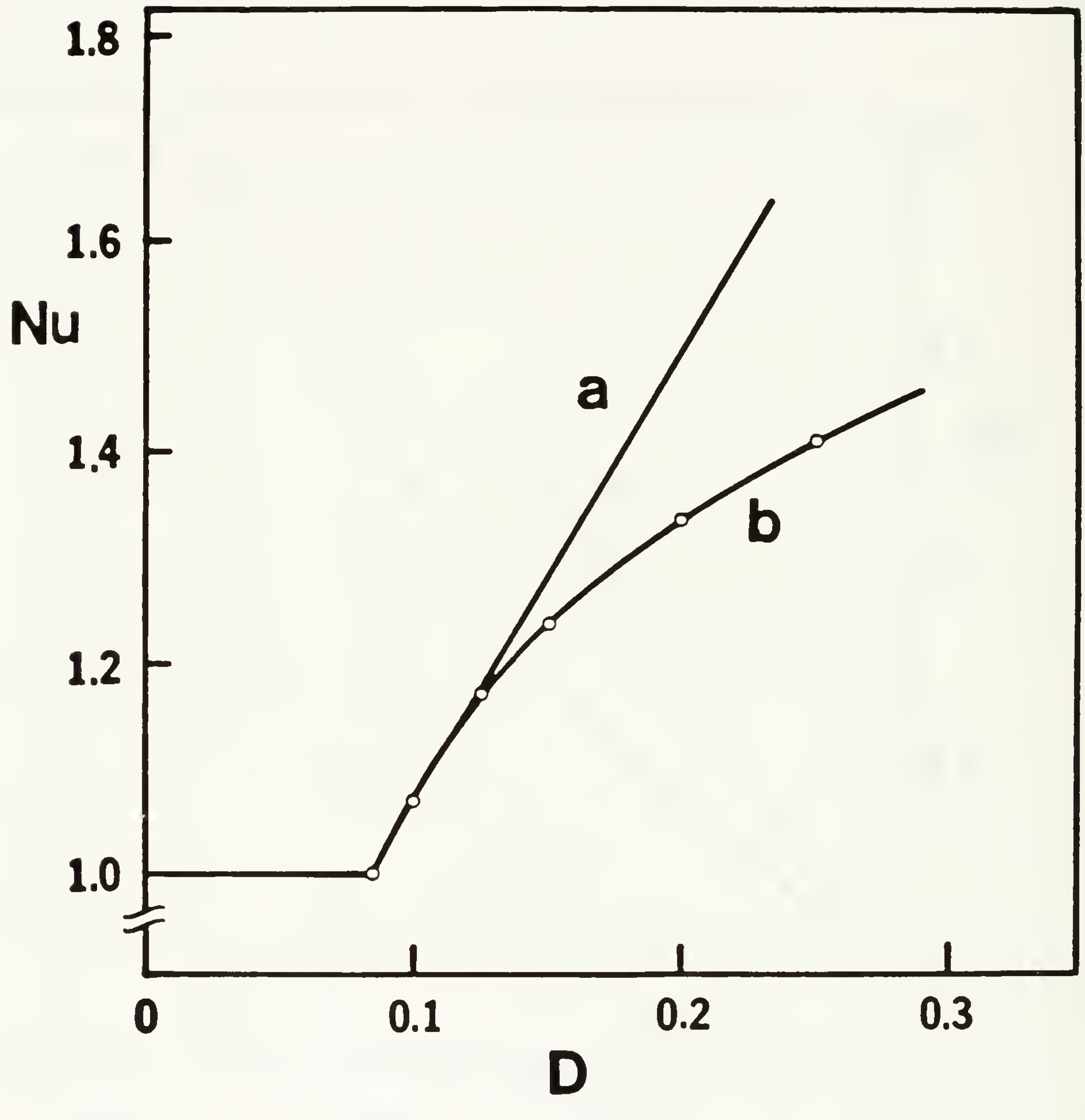

FIG. 5

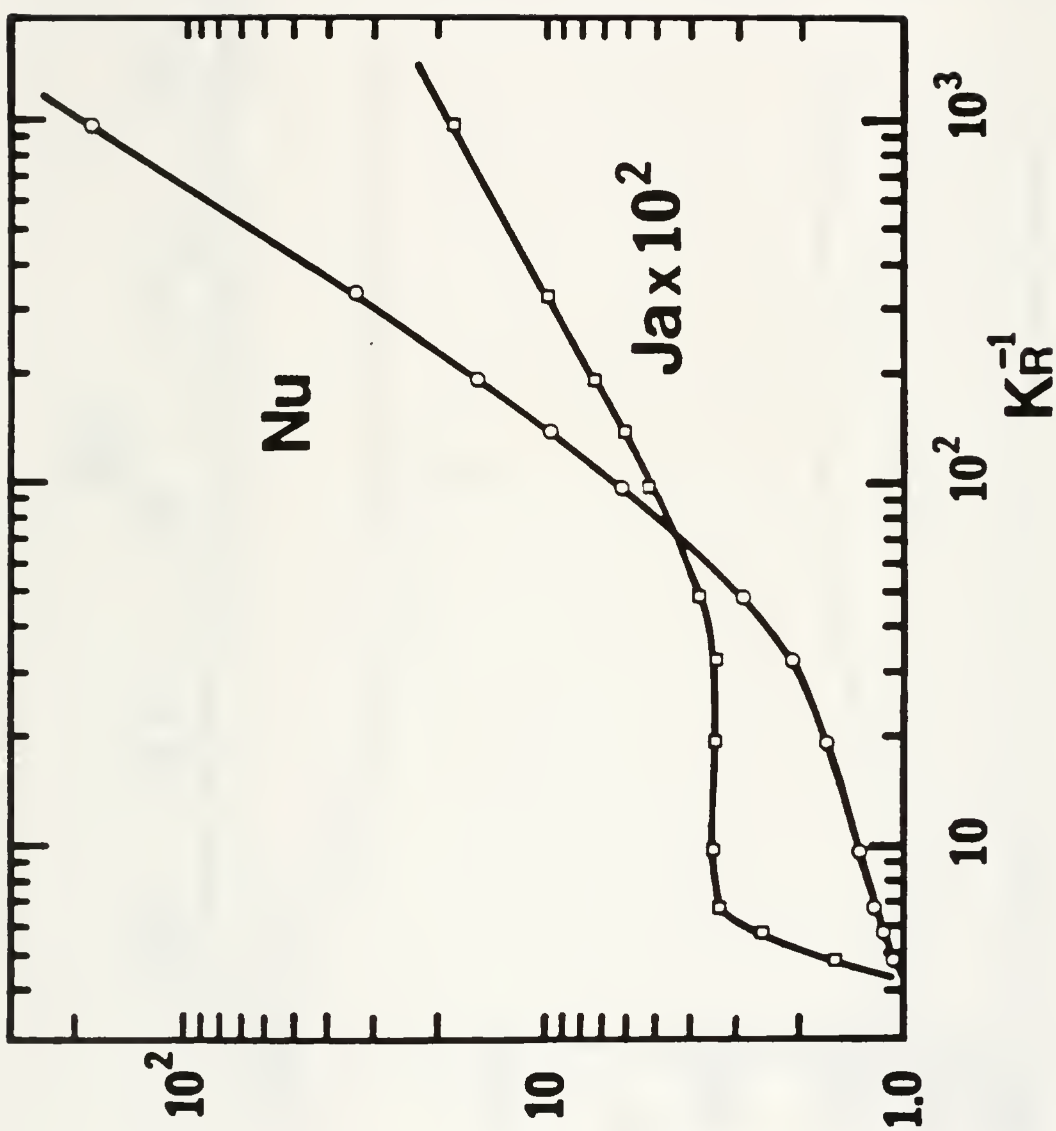

FIG 6

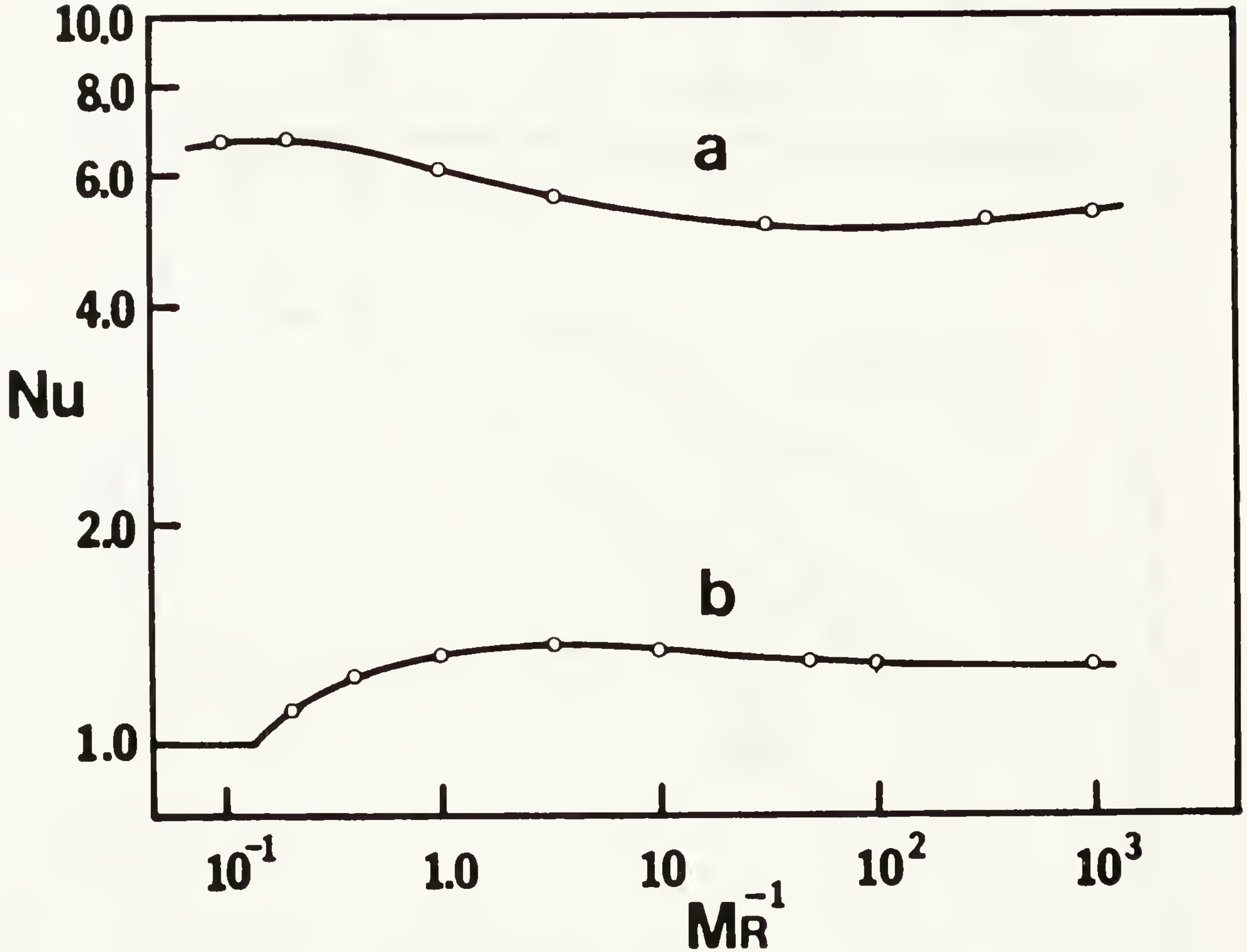

FIG. 7